

## ESTIMATIONS OF THE LEHMER MEAN BY THE HERON MEAN AND THEIR GENERALIZATIONS INVOLVING REFINED HEINZ OPERATOR INEQUALITIES

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ABSTRACT. As generalizations of the arithmetic and the geometric means, for positive real numbers  $a$  and  $b$ , the power difference mean  $J_q(a, b) = \frac{q}{q+1} \frac{a^{q+1} - b^{q+1}}{a^q - b^q}$ , the Lehmer mean  $L_q(a, b) = \frac{a^{q+1} + b^{q+1}}{a^q + b^q}$  and the Heron mean  $K_q(a, b) = (1 - q)\sqrt{ab} + q\frac{a+b}{2}$  are well known.

In this paper, concerning our recent results on estimations of the power difference mean, we obtain the greatest value  $\alpha = \alpha(q)$  and the least value  $\beta = \beta(q)$  such that the double inequality for the Lehmer mean

$$K_\alpha(a, b) < L_q(a, b) < K_\beta(a, b)$$

holds for any  $q \in \mathbb{R}$ . We also obtain an operator version of this estimation. Moreover, we discuss generalizations of the results on estimations of the power difference and the Lehmer means. This argument involves refined Heinz operator inequalities by Liang and Shi.

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