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THE COMPACTNESS OF A CLASS OF RADIAL OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. In this paper, we study some connection between the compactness of radial operators and the boundary behavior of the corresponding Berezin transform on weighted Bergman spaces. More precisely, we prove that, under some mild condition, the vanishing of the Berezin transform on the unit circle is equivalent to the compactness of a class of radial operators on weighted Bergman spaces. Moreover, we also study the radial essential commutant of the Toeplitz operator T_z.

1. Introduction and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and dm the normalized Lebesgue area measure on $\mathbb{D}$. For $\alpha > -1$, the weighted Bergman space $\mathcal{A}^2_\alpha$ is the space of all analytic functions on $\mathbb{D}$ which are in $L^2(\mathbb{D}, dm_\alpha)$, where $dm_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dm(z)$. For any nonnegative integer n, let $e_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^n$, $z \in \mathbb{D}$, here $\Gamma(\cdot)$ is the usual gamma function. It is easy to check that $\{e_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathcal{A}^2_\alpha$. It follows that $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in \mathcal{A}^2_\alpha$ if and only if $\|f\|^2 = \sum\limits_{n=0}^{\infty} \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} |a_n|^2 < \infty$. For $z \in \mathbb{D}$, $K^\alpha_z(w) = \frac{1}{(1-\bar{z}w)^{2+\alpha}}$ is the

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weighted Bergman reproducing kernel, that is, $f(z) = \langle f, K_z^\alpha \rangle$ for every $f \in \mathcal{A}_\alpha^2$, where $\langle \cdot, \cdot \rangle$ denotes the natural inner product in $\mathcal{A}_\alpha^2$.

For a bounded linear operator A on $\mathcal{A}_\alpha^2$, the Berezin transform of A is the function $\widetilde{A}$ on $\mathbb{D}$ defined by $\widetilde{A}(z) = \langle A k_z^\alpha, k_z^\alpha \rangle$, where k_z^α is the normalized reproducing kernel, i.e., $k_z^\alpha(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}, \quad w \in \mathbb{D}$. It is easy to see that each bounded operator on $\mathcal{A}_\alpha^2$ is uniquely determined by its Berezin transform. Thus, the behavior of an operator can be analyzed by exploring the corresponding Berezin transform. The important tool in study of the properties (such as the similarity and the invariant subspaces) of Toeplitz operators is the Berezin transform and the Mellin transform, we mention here that the papers (see [1, 2, 3, 6, 7, 9, 11, 12, 13]). We also notice that the Berezin transform of an operator A on $\mathcal{A}_\alpha^2$ has an explicit formula. In fact,

$$\begin{aligned}
\widetilde{A}(z) = \langle A k_z^\alpha, k_z^\alpha \rangle &= (1-|z|^2)^{2+\alpha} \langle A \frac{1}{(1-\bar{z}w)^{2+\alpha}}, \frac{1}{(1-\bar{z}w)^{2+\alpha}} \rangle \\
&= (1-|z|^2)^{2+\alpha} \langle A \sum_{n=0}^\infty \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} \bar{z}^n w^n, \sum_{m=0}^\infty \frac{\Gamma(m+2+\alpha)}{m!\Gamma(2+\alpha)} \bar{z}^m w^m \rangle \\
&= (1-|z|^2)^{2+\alpha} \sum_{m,n=0}^\infty \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} \sqrt{\frac{\Gamma(m+2+\alpha)}{m!\Gamma(2+\alpha)}} \langle A e_n(w), e_m(w) \rangle \bar{z}^n z^m
\end{aligned}$$

for all $z \in \mathbb{D}$. We know more about the boundary values of the Berezin transform in the case that the corresponding operator is compact. Since $\{k_z^\alpha\}$ converges weakly and uniformly to 0 as $|z|$ goes to 1, $\widetilde{A}(z)$ converges to 0 uniformly as $|z|$ approaches to 1, for any compact operator A on $\mathcal{A}_\alpha^2$. The main topic in this paper is to consider the inverse problem of determining for which operators on $\mathcal{A}_\alpha^2$ the vanishing on $\partial\mathbb{D}$ of the Berezin transform of the operator is equivalent to the compactness of the inducing operator. We know that this is not true for general case indicated by the following examples.

One example is the composition operator $C_\phi f = f \circ \phi$ induced by $\phi(z) = -z$, which is bounded on $\mathcal{A}_\alpha^2$ such that $\lim_{|z| \to 1^-} \widetilde{C_\phi}(z) = 0$, but C_ϕ is not compact. In fact, for any $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{A}_\alpha^2$, $\|C_\phi f\|^2 = \| \sum_{n=0}^\infty a_n (-z)^n \|^2 = \sum_{n=0}^\infty \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} |a_n|^2 = \|f\|^2$. It follows that C_ϕ is an isometry on $\mathcal{A}_\alpha^2$. So C_ϕ is not compact. However,

$$\begin{aligned}
\widetilde{C_\phi}(z) = \langle C_\phi k_z^\alpha, k_z^\alpha \rangle &= (1-|z|^2)^{2+\alpha} \langle C_\phi K_z^\alpha, K_z^\alpha \rangle \\
&= (1-|z|^2)^{2+\alpha} \langle K_z^\alpha(\phi), K_z^\alpha \rangle = (1-|z|^2)^{2+\alpha} K_z^\alpha(\phi(z)) \\
&= \frac{(1-|z|^2)^{2+\alpha}}{(1+|z|^2)^{2+\alpha}} \to 0 \quad (\mathit{as}\ |z| \to 1^-).
\end{aligned}$$

Another example of a non-compact operator with boundary vanishing Berezin transform is the projection operator $\mathcal{P}$ onto the closed subspace generated by the orthonormal set $\{e_{2^n} : n = 0, 1, 2, \dots\}$. In fact, since $\mathcal{P}$ has an infinite dimensional range, it is not compact, but the Berezin transform

$$\begin{aligned}\widetilde{\mathcal{P}}(z) &= (1-|z|^2)^{2+\alpha}\sum_{m,n=0}^{\infty}\sqrt{\tfrac{\Gamma(2^n+2+\alpha)}{2^n!\Gamma(2+\alpha)}}\sqrt{\tfrac{\Gamma(2^m+2+\alpha)}{2^m!\Gamma(2+\alpha)}}\langle Pe_{2^n}, e_{2^m}\rangle \bar{z}^{2^n} z^{2^m}\\ &= (1-|z|^2)^{2+\alpha}\sum_{n=0}^{\infty}\tfrac{\Gamma(2^n+2+\alpha)}{2^n!\Gamma(2+\alpha)}(|z|^2)^{2^n} \to 0 \ \ (\ as\ |z| \to 1^-).\end{aligned}$$

Our main results show that for a class of radial operators, under some mild condition, the compactness is equivalent to having vanishing Berezin transform on $\partial\mathbb{D}$. Moreover, we also give a characterization of the radial essential commutant of the Toeplitz operator T_z on $\mathcal{A}^2_\alpha$.

2. Background

Let $\mathfrak{L}(\mathcal{A}^2_\alpha)$ denote the operator algebra of all bounded linear operators on $\mathcal{A}^2_\alpha$ and

$$\vartheta = \{A \in \mathfrak{L}(\mathcal{A}^2_\alpha)|\ \widetilde{A}(z) \longrightarrow 0 \text{ (as } |z| \longrightarrow 1^-) \text{ implies that } A \text{ is compact}\}.$$

The previous examples in the last section show that ϑ is a proper subset of $\mathfrak{L}(\mathcal{A}^2_\alpha)$. The problem of determining if this is true even for a single Toeplitz operator has been opened for a number of years.

Recall that for $f \in L^\infty(\mathbb{D})$, the Toeplitz operator T_f with symbol f is the operator on $\mathcal{A}^2_\alpha$ defined by $T_f g = P(fg)$, where P is the orthogonal projection from $L^2(D, dm_\alpha)$ onto $\mathcal{A}^2_\alpha$. That is, $(T_f g)(z) = (P(fg))(z) = \int_{\mathbb{D}} f(w)g(w)K^\alpha_w(z)dm_\alpha(w)$. The compactness of Toeplitz operators on the weighted Bergman space have been of interest to many mathematicians. Zhu (see [5]) and Luecking (see [6]) proved that T_f is in ϑ whenever f is positive. Korenblum and Zhu (see[9]) proved that the same conclusion holds if f is radial. Axler and Zheng (see [1]) proved, for any $f \in L^\infty(\mathbb{D})$, $T_f \in \vartheta$. Actually, they showed that finite sums of finite products of Toeplitz operators with $L^\infty(\mathbb{D})$ symbols belong to the class ϑ. Zorboska (see [2]) and Stroethoff (see [12]) proved that for a class of radial operators, the compactness of an operator is equivalent to the vanishing of the Berezin transform on the unit circle. Following this direction, we generalize the corresponding results to the weighted Bergman spaces.

For $f \in L^1(\mathbb{D})$ we define $\mathrm{rad}(f)$ by

$$\mathrm{rad}(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt.$$

We say that $\mathrm{rad}(f)$ is the radialization of f, and f is radial if it is equal to its radialization. Thus a function is radial if and only if $f(z) = f(|z|)$. We also call the function $\widetilde{f(z)} = \int_{\mathbb{D}} f(w)|k^\alpha_z(w)|^2 dm_\alpha(w)$ is the Berezin transform of f, which is a weighted average of f. For a bounded operator A on $\mathcal{A}^2_\alpha$, we defined $\mathrm{Rad}(A)$ by

$$\mathrm{Rad}(A) = \frac{1}{2\pi}\int_0^{2\pi} U_t^* A U_t\, dt,$$

which means that for any f and g in $\mathcal{A}^2_\alpha$,

$$\langle \mathrm{Rad}(A)f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle U_t^* A U_t f, g\rangle dt$$

where U_t is the unitary operator such that $(U_t f)(z) = f(e^{-it}z)$ for f in $\mathcal{A}^2_\alpha$ and z in $\mathbb{D}$. We say that the operator A is radial if $A = \mathrm{Rad}(A)$.

For our main results, we need the following lemmas.

Lemma 2.1. *For any $A \in \mathcal{L}(\mathcal{A}^2_\alpha)$ and $f \in L^1(\mathbb{D})$,*
(i) $\widetilde{\mathrm{Rad}(A)} = \mathrm{rad}(\widetilde{A})$. In addition, A is radial if and only if $\widetilde{A}$ is radial .
(ii) $\widetilde{\mathrm{rad}(f)} = \mathrm{rad}(\widetilde{f})$. In addition, f is radial if and only if $\widetilde{f}$ is radial. Especially, T_f is radial if and only if f is radial.

Proof. Since

$$U_t k_z^\alpha(w) = k_z^\alpha(e^{-it}w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}e^{-it}w)^{2+\alpha}} = \frac{(1-|ze^{it}|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{ze^{it}}w)^{2+\alpha}} = k^\alpha_{e^{it}z}(w),$$

then

$$\begin{aligned}\widetilde{\mathrm{Rad}(A)}(z) &= \langle \mathrm{Rad}(A)k_z^\alpha, k_z^\alpha\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle U_t^*AU_t k_z^\alpha, k_z^\alpha\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\widetilde{A}(e^{it}z)dt = \mathrm{rad}(\widetilde{A})(z).\end{aligned}$$

Hence, if A is a radial operator, then $\widetilde{A}(z) = \mathrm{rad}(\widetilde{A})(z)$. That is, $\widetilde{A}$ is a radial function. On the other hand, if $\mathrm{rad}(\widetilde{A})(z) = \widetilde{A}(z)$, then $\widetilde{\mathrm{Rad}(A)}(z) = \widetilde{A}(z)$. Since the operators are uniquely determined by their Berezin transform, so $A = \mathrm{Rad}(A)$.

For the second term, since

$$\begin{aligned}\widetilde{\mathrm{rad}(f)}(z) &= \int_{\mathbb{D}}\mathrm{rad}(f)(w)|k_z^\alpha(w)|^2 dm_\alpha(w) = \int_{\mathbb{D}}\frac{1}{2\pi}\int_0^{2\pi} f(e^{it}w)dt|k_z^\alpha(w)|^2dm_\alpha(w) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{D}} f(u)|k^\alpha_{e^{it}z}(u)|^2 dm_\alpha(u)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\widetilde{f(e^{it}z)}\,dt = \mathrm{rad}(\widetilde{f})(z).\end{aligned}$$

Thus, f is radial if and only if $\widetilde{f}$ is radial if and only if T_f is radial. □

The following lemma says that every radial operator on $\mathcal{A}^2_\alpha$ actually is a diagonal operator.

Lemma 2.2. *Let A be a radial bounded operator on $\mathcal{A}^2_\alpha$. Then A is a diagonal operator with respect to the orthonormal basis $\{e_n\}$ of $\mathcal{A}^2_\alpha$.*

Proof. Since A is a radial bounded operator on $\mathcal{A}^2_\alpha$, then $A = \mathrm{Rad}(A)$. We have

$$\begin{aligned}\langle Ae_n, e_m\rangle &= \langle \mathrm{Rad}(A)e_n, e_m\rangle = \langle \frac{1}{2\pi}\int_0^{2\pi} U_t^*AU_t\ e_n, e_m\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n-m)t}\langle Ae_n, e_m\rangle dt = 0, \text{ for } n \neq m.\end{aligned}$$

$\square$

3. Main results

Note that an operator A on weighted Bergman spaces is compact if and only if A^*A is compact if and only if there exists a unitary operator U such that U^*A^*AU is a compact radial operator. In the sequel, suppose that A is a radial operator on $\mathcal{A}^2_\alpha$ and $a_n = \langle Ae_n, e_n\rangle$, we want to explore the relation between the behavior of the sequence $\{a_n\}$ and the boundary behavior of the Berezin transform $\widetilde{A}$. Of course, if $\lim\limits_{n\to\infty} a_n = 0$, then $\widetilde{A}(z)$ converges to 0 (as $|z| \to 1^-$). The converse is not always true as the two counterexamples of diagonal non-compact operators in Section 1. We will show that, under some mild restriction on the sequence $\{a_n\}$, $\widetilde{A}(z) \longrightarrow 0$ (as $|z| \longrightarrow 1^-$) is a sufficient condition for the compactness of A.

For a radial operator A, $\widetilde{A}(z) = (1-|z|^2)^{2+\alpha}\sum\limits_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n|z|^{2n}$. Thus the question of whether A is in ϑ is equivalent to the following problem: when does $\lim\limits_{|z|\longrightarrow 1^-}(1-|z|^2)^{2+\alpha}\sum\limits_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n|z|^{2n} = 0$ imply that $a_n \longrightarrow 0$ (as $n \longrightarrow \infty$)?

Lemma 3.1. (*see* [4, 8]) *Suppose* $\lambda \geq 0$ *and* $\lim\limits_{t\longrightarrow 1^-}(1-t)^\lambda \sum\limits_{n=0}^{\infty} b_n t^n = 0$.

If $\sup\limits_{n\geq 0} n^{1-\lambda}|b_n| \leq \infty$, *then* $\lim\limits_{n\longrightarrow\infty}\frac{\sum_{k=0}^{n} b_k}{(n+1)^\lambda} = 0$.

Lemma 3.2. *Suppose that* $f \in L^\infty[0,1)$ *and* $a_n(f) = \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}\int_0^1 f(r)r^{2n+1}(1-r^2)^\alpha dr$, $n \geq 0$. *Then* $a_n(f) \longrightarrow 0$ (*as* $n \longrightarrow \infty$) *if and only if*

$$\lim_{t\longrightarrow 1^-}(1-t)^{2+\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n = 0.$$

Proof. First, suppose that $a_n(f) \to 0$ (as $n \to \infty$). For any $\varepsilon > 0$, there exists a positive integer N, when $n > N$, we have $|a_n(f)| < \frac{\varepsilon}{2}$. For the $\varepsilon > 0$ above, there exists $0 < \delta < 1$, when $1-\delta < t < 1$, we obtain

$$\begin{aligned}
&|(1-t)^{2+\alpha}\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n| \\
\leq\ & |(1-t)^{2+\alpha}\sum_{n=0}^{N}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n| + |(1-t)^{2+\alpha}\sum_{n=N+1}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n| \\
<\ & \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}|(1-t)^{2+\alpha}\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}t^n| < \varepsilon.
\end{aligned}$$

Thus $\lim\limits_{t\longrightarrow 1^-}(1-t)^{2+\alpha}\sum\limits_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n = 0$.

For the other implication, assume that $\lim\limits_{t\longrightarrow 1^-}(1-t)^{2+\alpha}\sum\limits_{n=0}^{\infty}\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n = 0$.

Note that

$$\begin{aligned}
&(1-t)^{2+\alpha}\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n=(1-t)^{1+\alpha}\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)(t^n-t^{n+1})\\
&\quad=(1-t)^{1+\alpha}[\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n-\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^{n+1}]\\
&\quad=(1-t)^{1+\alpha}[a_0(f)+\sum_{n=1}^{\infty}\tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)t^n-\sum_{n=1}^{\infty}\tfrac{\Gamma(n+1+\alpha)}{(n-1)!\Gamma(2+\alpha)}a_{n-1}(f)t^n].
\end{aligned}$$

Let $b_0=a_0(f), b_n=\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)-\frac{\Gamma(n+1+\alpha)}{(n-1)!\Gamma(2+\alpha)}a_{n-1}(f)$. In the sequel, we often use the same letter M, depending only on the allowed parameters, to denote various positive constants which may change at each occurrence. By the hypothesis, $|f(z)|\le M<\infty$ a.e. and

$$\begin{aligned}
|b_n|&=\Big|\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f)-\frac{\Gamma(n+1+\alpha)}{(n-1)!\Gamma(2+\alpha)}a_{n-1}(f)\Big|\\
&=\Big|\frac{\Gamma^2(n+2+\alpha)}{(n!)^2\Gamma^2(2+\alpha)}\int_0^1 f(r)r^{2n+1}(1-r^2)^\alpha dr\\
&\quad-\frac{\Gamma^2(n+1+\alpha)}{((n-1)!)^2\Gamma^2(2+\alpha)}\int_0^1 f(r)r^{2n-1}(1-r^2)^\alpha dr\Big|\\
&=\frac{\Gamma^2(n+1+\alpha)}{((n-1)!)^2\Gamma^2(2+\alpha)}\Big|\frac{n^2+2n(1+\alpha)+\ (1+\alpha)^2}{n^2}\int_0^1 f(r)r^{2n+1}(1-r^2)^\alpha dr\\
&\quad-\int_0^1 f(r)r^{2n-1}(1-r^2)^\alpha dr\Big|\\
&=\frac{\Gamma^2(n+1+\alpha)}{((n-1)!)^2\Gamma^2(2+\alpha)}\Big|\int_0^1 f(r)r^{2n-1}(1-r^2)^\alpha(r^2-1)dr\\
&\quad+\frac{2n(1+\alpha)+(1+\alpha)^2}{n^2}\int_0^1 f(r)r^{2n+1}(1-r^2)^\alpha dr\Big|\\
&\le\frac{\Gamma^2(n+1+\alpha)}{((n-1)!)^2\Gamma^2(2+\alpha)}\Big|\int_0^1 f(r)r^{2n-1}(1-r^2)^\alpha(r^2-1)dr\Big|\\
&\quad+\frac{\Gamma^2(n+1+\alpha)}{(n!)^2\Gamma^2(2+\alpha)}\Big|(2n(1+\alpha)+(1+\alpha)^2)\int_0^1 f(r)r^{2n+1}(1-r^2)^\alpha dr\Big|\\
&\le\frac{\Gamma^2(n+1+\alpha)}{((n-1)!)^2\Gamma^2(2+\alpha)}M\Big|\int_0^1 r^{2n-1}(1-r^2)^\alpha(r^2-1)dr\Big|\\
&\quad+\frac{\Gamma^2(n+1+\alpha)}{(n!)^2\Gamma^2(2+\alpha)}(2n(1+\alpha)+(1+\alpha)^2)M\Big|\int_0^1 r^{2n+1}(1-r^2)^\alpha dr\Big|\\
&=\frac{\Gamma^2(n+1+\alpha)}{((n-1)!)^2\Gamma^2(2+\alpha)}M\frac{\Gamma(n)\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}\\
&\quad+\frac{\Gamma^2(n+1+\alpha)}{(n!)^2\Gamma^2(2+\alpha)}(2n(1+\alpha)+(1+\alpha)^2)M\frac{\Gamma(n+1)\Gamma(1+\alpha)}{\Gamma(n+2+\alpha)}\\
&=\frac{M\Gamma(n+1+\alpha)}{n!\Gamma(2+\alpha)}\frac{3n+1+\alpha}{n+1+\alpha}.
\end{aligned}$$

By Stirling's formula $\frac{\Gamma(n+1)}{\Gamma(1+n+\mu)} \sim \frac{1}{n^{\mu}}(\mu > 0)$, we have $\frac{|b_n|}{n^{\alpha}} \leq M < \infty$. Using Lemma 3.1, we have $\lim\limits_{n\longrightarrow\infty} \frac{\sum\limits_{k=0}^{n} b_k}{(n+1)^{1+\alpha}} = 0$. Again

$$\begin{aligned}\sum_{k=0}^{n} b_k &= b_0 + \sum_{k=1}^{n} b_k \\ &= a_0(f) + \sum_{k=1}^{n}\left(\tfrac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}a_k(f) - \tfrac{\Gamma(k+1+\alpha)}{(k-1)!\Gamma(2+\alpha)}a_{k-1}(f)\right) \\ &= \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n(f).\end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \frac{\sum\limits_{k=0}^{n} b_k}{(n+1)^{1+\alpha}} = \lim_{n\longrightarrow\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)(n+1)^{1+\alpha}}a_n(f) = \lim_{n\longrightarrow\infty} \frac{a_n(f)}{\Gamma(2+\alpha)} = 0.$$

So $\lim\limits_{n\longrightarrow\infty} a_n(f) = 0$. □

Lemma 3.3. (*see* [10]) *Let $\mathcal{H}$ be a separable Hilbert space with basis $\{e_n\}$, and $\{a_n\}$ a complex sequence such that $M = \sup\{|a_n| : n \geq 1\} < \infty$. If $Ae_n = a_n e_n$ for all n, then A extends by linearity to a bounded operator on $\mathcal{H}$ with $\|A\| = M$. Moreover, A is compact if and only if $a_n \longrightarrow 0$ (as $n \longrightarrow \infty$).*

In [12], Stroethoff proved that: If f is a bounded and uniformly continuous with respect to the Bergman metric on the unit ball B_n, then T_f is compact on $\mathcal{A}^2_\alpha(B_n)$ if and only if $\widetilde{f(z)} \longrightarrow 0$ as $|z| \longrightarrow 1^-$. In the following, we give a sufficient condition for a general radial operator A such that the operator A is compact.

Theorem 3.4. *Let A be a bounded radial operator on $\mathcal{A}^2_\alpha$ with diagonal $\{a_n\}$, with respect to the orthonormal basis $\{e_n\}$, such that $n(a_n - a_{n-1})$ is bounded. Then A belongs to the class ϑ.*

Proof. Since A is radial,

$$\begin{aligned}\widetilde{A}(z) &= (1-|z|^2)^{2+\alpha} \sum_{n=0}^{\infty} \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} a_n|z|^{2n} \\ &= (1-|z|^2)^{1+\alpha}[a_0 + \sum_{n=1}^{\infty} \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} a_n|z|^{2n} - \sum_{n=1}^{\infty} \tfrac{\Gamma(n+1+\alpha)}{(n-1)!\Gamma(2+\alpha)} a_{n-1}|z|^{2n}] \\ &= (1-|z|^2)^{1+\alpha} \sum_{n=0}^{\infty} b_n|z|^{2n},\end{aligned}$$

where $b_0 = a_0$ and $b_n = \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}a_n - \frac{\Gamma(n+1+\alpha)}{(n-1)!\Gamma(2+\alpha)}a_{n-1} = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(2+\alpha)}[n(a_n - a_{n-1}) + (1+\alpha)a_n]$. Since both $n(a_n - a_{n-1})$ and $(1+\alpha)a_n$ are bounded, we have from Stirling's formula that $\frac{b_n}{n^\alpha} = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(2+\alpha)n^\alpha}[n(a_n - a_{n-1}) + (1+\alpha)a_n]$ is bounded. From Lemma 3.1, we get $\lim\limits_{n\to\infty} \frac{\sum_{k=0}^{n} b_k}{(n+1)^{1+\alpha}} = 0$. By Lemma 3.2, we have $\lim\limits_{n\longrightarrow\infty} a_n = 0$. According to Lemma 3.3, the operator A is compact. □

Lemma 3.5. (*see* [9]) *Let* $K_1,\ K_2 \in L^1[0,+\infty)$. *If* $\int_0^{+\infty} K_j(t)\, t^{ix} dt \neq 0$ *for* $j = 1, 2$ *and all real number* x, *then for any* $g \in L^\infty[0,+\infty)$, $\lim\limits_{\varepsilon \longrightarrow 0^+} \int_0^{+\infty} K_1(t)\, g(\varepsilon t) dt = 0$ *if and only if* $\lim\limits_{\varepsilon \longrightarrow 0^+} \int_0^{+\infty} K_2(t)\, g(\varepsilon t) dt = 0$.

Theorem 3.6. *Let* f *be a bounded radial function on* $\mathbb{D}$. *Then the following conditions are equivalent:*

(*i*) $T_f : \mathcal{A}^2_\alpha \longrightarrow \mathcal{A}^2_\alpha$ *is compact.*

(*ii*) $\widetilde{f(z)} \longrightarrow 0$ *as* $|z| \longrightarrow 1^-$.

(*iii*) $\frac{1}{(1-x)^{1+\alpha}} \int_x^1 f(\sqrt{t})(1-t)^\alpha dt \longrightarrow 0$ *as* $x \longrightarrow 1^-$.

Proof. Since f is a bounded radial function on $\mathbb{D}$, then T_f is radial.

For $(i) \Longrightarrow (ii)$, since T_f: $\mathcal{A}^2_\alpha \longrightarrow \mathcal{A}^2_\alpha$ is compact, then $d_n(f) = \langle T_f e_n, e_n \rangle \to 0$ (as $n \to \infty$). Since

$$\widetilde{f(z)} = \widetilde{T_f}(z) = \langle T_f k^\alpha_z, k^\alpha_z \rangle = (1-|z|^2)^{2+\alpha} \sum_{n=0}^{\infty} \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} d_n(f) |z|^{2n}.$$

By Lemma 3.2, we have $\lim\limits_{|z| \longrightarrow 1^-} (1-|z|)^{2+\alpha} \sum\limits_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} d_n(f)|z|^n = 0$. That is, $\lim\limits_{|z| \longrightarrow 1^-} \widetilde{f(z)} = 0$.

For $(ii) \Longrightarrow (iii)$, since $\lim\limits_{|z| \longrightarrow 1^-} \widetilde{f(z)} = 0$, then

$$\lim_{t \longrightarrow 1^-} (1-t)^{2+\alpha} \sum_{n=0}^{\infty} \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} d_n(f) t^n = 0.$$

By Lemma 3.2, we have $\lim\limits_{n \longrightarrow \infty} d_n(f) = 0$. Since

$$d_n(f) = \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t}) t^n (1-t)^\alpha dt,$$

we obtain $\lim\limits_{n \longrightarrow \infty} \frac{n^{1+\alpha}}{\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t})\, t^n (1-t)^\alpha dt = 0$ by Stirling's formula $\frac{\Gamma(n+2+\alpha)}{\Gamma(n+1)} \sim n^{1+\alpha}$. Let $1 - t = \frac{u}{n}$, then

$$\begin{aligned} &\tfrac{n^{1+\alpha}}{\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t})\, t^n (1-t)^\alpha dt \\ &\quad= \tfrac{1}{\Gamma(1+\alpha)} \int_0^n f(\sqrt{1-\tfrac{u}{n}})\, (1-\tfrac{u}{n})^n u^\alpha du \\ &\quad= \tfrac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} g(\varepsilon u) K_\varepsilon(u) du, \end{aligned}$$

where $\varepsilon = \frac{1}{n}$ and

$$g(u) = \begin{cases} f(\sqrt{1-u}), & 0 \leq u \leq 1, \\ 0, & u > 1. \end{cases}$$

$$K_\varepsilon(u) = \begin{cases} (1-\varepsilon u)^{\frac{1}{\varepsilon}} u^\alpha, & 0 \leq u \leq \frac{1}{\varepsilon}, \\ 0, & u > \frac{1}{\varepsilon}. \end{cases}$$

Since $0 \leq K_\varepsilon(u) \leq e^{-u}\, u^\alpha$, the dominated convergence theorem implies that $K_\varepsilon(u) \longrightarrow e^{-u} u^\alpha$ in $L^1[0,+\infty)$ as $\varepsilon \longrightarrow 0^+$. Therefore, the condition $\lim\limits_{n \longrightarrow +\infty} \frac{n^{1+\alpha}}{\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t}) t^n (1-t)^\alpha dt = 0$ is equivalent to $\lim\limits_{\varepsilon \longrightarrow 0^+} \int_0^{+\infty} e^{-u} g(\varepsilon u) u^\alpha du =$

0. Similarly, $\lim\limits_{x\longrightarrow 1^-} \frac{\int_x^1 f(\sqrt{t})(1-t)^\alpha dt}{(1-x)^{1+\alpha}} = 0$ is equivalent to $\lim\limits_{\varepsilon\longrightarrow 0^+} \int_0^{+\infty} \chi_{[0,1]}(u)u^\alpha\, g(\varepsilon\, u)du = 0$. The desired result follows from Lemma 3.5.

For $(iii) \Longrightarrow (i)$, if $\lim\limits_{x\longrightarrow 1^-} \frac{\int_x^1 f(\sqrt{t})\,(1-t)^\alpha dt}{(1-x)^{1+\alpha}} = 0$, note that

$$d_n(f) = \frac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t})(1-t)^\alpha t^n dt,$$

we need only to prove $\lim\limits_{n\longrightarrow\infty} d_n(f) = 0$. Set $h(x) = \frac{\int_x^1 f(\sqrt{t})\,(1-t)^\alpha dt}{(1-x)^{1+\alpha}}$, $x \in [0,1)$, then

$$\begin{aligned}
&\frac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} n \int_0^1 h(x)(1-x)^{1+\alpha} x^{n-1} dx \\
&\quad= \frac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} n \int_0^1 \frac{\int_x^1 f(\sqrt{t})\,(1-t)^\alpha dt}{(1-x)^{1+\alpha}} (1-x)^{1+\alpha} x^{n-1} dx \\
&\quad= \frac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} n \int_0^1 f(\sqrt{t})\,(1-t)^\alpha dt \int_0^t x^{n-1} dx \\
&\quad= \frac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t})\,(1-t)^\alpha t^n dt.
\end{aligned}$$

The second equality follows from the Fubinis's theorem. This gives $\lim\limits_{n\longrightarrow\infty} d_n(f) = 0$. By Lemma 3.3, we get the compactness of T_f. $\square$

4. Essential commutant of operator T_z

Let $\mathcal{I}(L^\infty)$ be the Toeplitz algebra generated by $\{T_f : f \in L^\infty(\mathbb{D})\}$. $\mathcal{I}(L^\infty)$ is the closed subalgebra of $\mathfrak{L}(\mathcal{A}_\alpha^2)$. We recall that the essential commutant of an operator $T \in \mathfrak{L}(\mathcal{A}_\alpha^2)$ is the set $C_e(T) = \{S \in \mathfrak{L}(\mathcal{A}_\alpha^2) : TS - ST \text{ is compact}\}$. Englis (see [3]) proved that

(a) $C_e(T_z) = \{S \in \mathfrak{L}(\mathcal{A}_\alpha^2) : S - T_z{}^* S T_z$ *is compact*$\}$,

(b) $C_e(T_z)$ *is a* C^**-algebra,*

(c) $T_\phi \in C_e(T_z)$ *for every* $\phi \in L^\infty(\mathbb{D})$.

Let l^∞ be the Banach space of bounded complex sequences indexed from $n \geq 0$. Consider the linear subspaces $d_0 = \{\{z_n\} \in l^\infty : (z_n - z_{n-1}) \longrightarrow 0\}$ and $d_1 = \{\{z_n\} \in l^\infty : \{n(z_n - z_{n-1})\} \in l^\infty\}$. It is clear that d_0 is closed in l^∞ and $d_1 \subset d_0$.

In [13], Daniel proved the following result on Bergman space. Now as an application of our main results, we generalize the conclusion to the weighted Bergman space.

Theorem 4.1. *Let* $S \in \mathfrak{L}(\mathcal{A}_\alpha^2)$ *be a radial operator. Then*
(i) $S \in C_e(T_z)$ *if and only if* $\{\lambda_n(S)\} \in d_0$,
(ii) If $S \in \mathcal{I}(L^\infty)$, *then* $\{\lambda_n(S)\} \in d_1$, *where* $\{\lambda_n(S)\}$ *is the diagonal elements of* S.

Proof. (i) Since S is a radial operator, we have $Sz^k = \lambda_k(S)z^k$. Observe that

$$T_z^* z^k = \begin{cases} \frac{k}{k+1+\alpha} z^{k-1}, & if\ k > 0, \\ 0, & if\ k = 0. \end{cases}$$

We have

$$(S - T_z^* S T_z)z^k = \lambda_k(S)z^k - \lambda_{k+1}(S)z^k + \frac{1+\alpha}{k+2+\alpha}\lambda_{k+1}(S)z^k.$$

That is, $S - T_z^* S T_z$ is radial. Note that $S - T_z^* S T_z$ is compact if and only if $\lambda_k(S) - \lambda_{k+1}(S) + \frac{1+\alpha}{k+2+\alpha}\lambda_{k+1}(S) \longrightarrow 0$ (as $k \to \infty$). Hence $\lambda_k(S) \in d_0$.
(ii) Since $S = T_f$ is radial, we have

$$\lambda_n(T_f) = \langle T_f e_n, e_n\rangle = \frac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t}) t^n (1-t)^\alpha dt.$$

Thus

$$\begin{aligned}
&|\lambda_{n+1}(T_f) - \lambda_n(T_f)| \\
&= \left|\tfrac{\Gamma(n+3+\alpha)}{(n+1)!\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t}) t^{n+1}(1-t)^\alpha dt - \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \int_0^1 f(\sqrt{t}) t^n (1-t)^\alpha dt\right| \\
&\le \|f\|_\infty \int_0^1 \left|\tfrac{\Gamma(n+3+\alpha)}{(n+1)!\Gamma(1+\alpha)} t^{n+1}(1-t)^\alpha - \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} t^n (1-t)^\alpha\right| dt \\
&= \|f\|_\infty \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \int_0^1 \left|\tfrac{n+2+\alpha}{n+1}(t-1) + \tfrac{1+\alpha}{n+1}\right| t^n (1-t)^\alpha dt \\
&\le \|f\|_\infty \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \left(\tfrac{n+2+\alpha}{n+1} \int_0^1 t^n (1-t)^{\alpha+1} dt + \tfrac{1+\alpha}{n+1} \int_0^1 t^n (1-t)^\alpha dt\right) \\
&= \|f\|_\infty \tfrac{\Gamma(n+2+\alpha)}{n!\Gamma(1+\alpha)} \left(\tfrac{n+2+\alpha}{n+1} \cdot \tfrac{\Gamma(n+1)\Gamma(2+\alpha)}{\Gamma(n+3+\alpha)} + \tfrac{1+\alpha}{n+1} \cdot \tfrac{\Gamma(n+1)\Gamma(1+\alpha)}{\Gamma(n+2+\alpha)}\right) \\
&= 2\|f\|_\infty \tfrac{1+\alpha}{n+1}.
\end{aligned}$$

This gives that $\{\lambda_n(S)\} \in d_1$. □

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