

## BESOV–DUNKL SPACES CONNECTED WITH GENERALIZED TAYLOR FORMULA ON THE REAL LINE

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**ABSTRACT.** In the present paper, we define for the Dunkl translation operators on the real line, the Besov–Dunkl space of functions for which the remainder in the generalized Taylor’s formula has a given order. We provide characterization of these spaces by the Dunkl convolution.

### 1. INTRODUCTION

There are many ways to define the Besov spaces (see [5, 6, 8, 13]) and the Besov–Dunkl spaces (see [1, 2, 3, 9]). It is well known that Besov spaces can be described by means of differences using the modulus of continuity of functions and that they can be also defined, for instance in terms of convolutions with different kinds of smooth functions.

Inspired by the work of Löfström and Peetre (see [10]) where they described for generalized translations, the space of functions for which the remainder in the generalized Taylor’s formula has a given order, we define in this paper the Besov-type space of functions associated with the Dunkl operator on the real line, that we call Besov–Dunkl spaces of order  $k$  for  $k = 1, 2, \dots$ . These spaces generalize the Besov spaces defined in [1, 5] for the case  $k = 1$ . Before, we need to recall some results in harmonic analysis related to the Dunkl theory.

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For a real parameter  $\alpha > -\frac{1}{2}$ , the Dunkl operator on  $\mathbb{R}$  denoted by  $\Lambda_\alpha$ , is a differential-difference operator introduced in 1989 by C. Dunkl in [7]. This operator is associated with the reflection group  $\mathbb{Z}_2$  on  $\mathbb{R}$  and can be considered as a perturbation of the usual derivative by reflection part. The operator  $\Lambda_\alpha$  plays a major role in the study of quantum harmonic oscillators governed by Wigner's commutation rules (see [14]). The Dunkl kernel  $E_\alpha$  related to  $\Lambda_\alpha$  is used to define the Dunkl transform  $\mathcal{F}_\alpha$  which enjoys properties similar to those of the classical Fourier transform. The Dunkl kernel  $E_\alpha$  satisfies a product formula (see [15]). This allows us to define the Dunkl translation  $\tau_x$ ,  $x \in \mathbb{R}$ . As a result, we have the Dunkl convolution  $*_\alpha$  (see next section).

The classical Taylor formula with integral remainder was extended to the one dimensional Dunkl operator  $\Lambda_\alpha$  (see [12]): for  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $a \in \mathbb{R}$ , we have

$$\tau_x(f)(a) = \sum_{p=0}^{k-1} b_p(x) \Lambda_\alpha^p f(a) + R_k(x, f)(a), \quad x \in \mathbb{R} \setminus \{0\},$$

with  $R_k(x, f)(a)$  is the integral remainder of order  $k$  given by

$$R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_{k-1}(x, y) \tau_y(\Lambda_\alpha^k f)(a) A_\alpha(y) dy,$$

where  $\mathcal{E}(\mathbb{R})$  is the space of infinitely differentiable functions on  $\mathbb{R}$  and  $(\Theta_p)_{p \in \mathbb{N}}$ ,  $(b_p)_{p \in \mathbb{N}}$  are two sequences of functions constructed inductively from the function  $A_\alpha$  defined on  $\mathbb{R}$  by  $A_\alpha(x) = |x|^{2\alpha+1}$  (see next section).

Now, we introduce the following weighted function spaces: Let  $k = 1, 2, \dots$ ,  $0 < \beta < 1$ ,  $1 \leq p < +\infty$  and  $1 \leq q \leq +\infty$ .

• We denote by  $L^p(\mu_\alpha)$  the space of complex-valued functions  $f$ , measurable on  $\mathbb{R}$  such that

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < +\infty,$$

where  $\mu_\alpha$  is a weighted Lebesgue measure associated with the Dunkl operator (see next section).

• (Besov–Dunkl spaces of order  $k$ )  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha}$  denote the subspace of functions  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1} f$  are in  $L^p(\mu_\alpha)$  and satisfying

$$\int_0^{+\infty} \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty$$

and

$$\sup_{x>0} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} < +\infty \quad \text{if} \quad q = +\infty,$$

with  $\omega_{p,\alpha}^k(x, f) = \|R_{k-1}(x, f) + R_{k-1}(-x, f) - (b_{k-1}(x) + b_{k-1}(-x)) \Lambda_\alpha^{k-1} f\|_{p,\alpha}$ . Here we put for  $k = 1$ ,  $\Lambda_\alpha^0 f = f$ ,  $R_0(x, f) = \tau_x(f)$  and  $R_0(-x, f) = \tau_{-x}(f)$ .

• We put

$$\mathcal{A}_k = \left\{ \phi \in \mathcal{S}_*(\mathbb{R}) : \int_0^{+\infty} x^{2i} \phi(x) d\mu_\alpha(x) = 0, \forall i \in \{0, 1, \dots, [\frac{k-1}{2}]\} \right\},$$

where  $\mathcal{S}_*(\mathbb{R})$  is the space of even Schwartz functions on  $\mathbb{R}$  and  $[\frac{k-1}{2}]$  is the integer part of the number  $\frac{k-1}{2}$ . Let  $\phi \in \mathcal{A}_k$  (see Example 4.4, section 4), we shall denote by  $\mathcal{C}_{p,q,\phi}^{k,\beta,\alpha}$  the subspace of functions  $f$  in  $\mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{2i} f \in L^p(\mu_\alpha)$ ,  $0 \leq i \leq [\frac{k-1}{2}]$  and satisfying

$$\int_0^{+\infty} \left( \frac{\|f *_\alpha \phi_t\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} < +\infty \quad \text{if } q < +\infty$$

and  $\sup_{t>0} \frac{\|f *_\alpha \phi_t\|_{p,\alpha}}{t^{\beta+k-1}} < +\infty \quad \text{if } q = +\infty,$

where  $\phi_t$  is the dilation of  $\phi$  given by  $\phi_t(x) = \frac{1}{t^{2(\alpha+1)}} \phi(\frac{x}{t})$ , for all  $t \in (0, +\infty)$  and  $x \in \mathbb{R}$ .

Our aim in this paper is to generalize to the order  $k = 1, 2, \dots$ , the results obtained in [1, 5] for the case  $k = 1$ . For this purpose, we give some properties and estimates of the integral remainder of order  $k$  and we establish that

$$\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \cap L^p(\mu_\alpha) = \mathcal{C}_{p,q,\phi}^{k,\beta,\alpha}.$$

It's clear from this equality that  $\mathcal{C}_{p,q,\phi}^{k,\beta,\alpha}$  is independent of the specific selection of the fuction  $\phi$  in  $\mathcal{A}_k$ .

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with the Dunkl operator  $\Lambda_\alpha$ .

In section 3, we prove some properties and estimates of the integral remainder of order  $k$ . Finally, we establish the coincidence between the characterizations of the Besov–Dunkl spaces of order  $k$ .

Along this paper, we use  $c$  to represent a suitable positive constant which is not necessarily the same in each occurrence.

## 2. PRELIMINARIES

In this section, we recall some notations and results in Dunkl theory on  $\mathbb{R}$  and we refer for more details to [4, 7, 15].

The Dunkl operator is given for  $x \in \mathbb{R}$  by

$$\Lambda_\alpha f(x) = \frac{df}{dx}(x) + \frac{2\alpha + 1}{x} \left[ \frac{f(x) - f(-x)}{2} \right], \quad f \in \mathcal{C}^1(\mathbb{R}).$$

For  $\lambda \in \mathbb{C}$ , the initial problem

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},$$

has a unique solution  $E_\alpha(\lambda)$  called Dunkl kernel given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha + 1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where  $j_\alpha$  is the normalized Bessel function of the first kind and order  $\alpha$ . Let  $A_\alpha$  the function defined on  $\mathbb{R}$  by

$$A_\alpha(x) = |x|^{2\alpha+1}, \quad x \in \mathbb{R},$$

and  $\mu_\alpha$  the weighted Lebesgue measure on  $\mathbb{R}$  given by

$$d\mu_\alpha(x) = \frac{A_\alpha(x)}{2^{\alpha+1}\Gamma(\alpha + 1)}dx. \tag{2.1}$$

There exists an analogue of the classical Fourier transform with respect to the Dunkl kernel called the Dunkl transform and denoted by  $\mathcal{F}_\alpha$ . The Dunkl transform enjoys properties similar to those of the classical Fourier transform and is defined for  $f \in L^1(\mu_\alpha)$  by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} f(y) E_\alpha(-ixy) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

For all  $x, y, z \in \mathbb{R}$ , we consider

$$W_\alpha(x, y, z) = \frac{(\Gamma(\alpha + 1))^2}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} (1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x})\Delta_\alpha(x, y, z)$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in \mathbb{R} \setminus \{0\}, z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_\alpha(x, y, z) = \begin{cases} \frac{((|x|+|y|)^2-z^2)[z^2-(|x|-|y|)^2]^{\alpha-\frac{1}{2}}}{|xyz|^{2\alpha}} & \text{if } |z| \in S_{x,y} \\ 0 & \text{otherwise} \end{cases}$$

where

$$S_{x,y} = [||x| - |y||, |x| + |y|].$$

The kernel  $W_\alpha$ , is even and we have

$$W_\alpha(x, , y, z) = W_\alpha(y, x, z) = W_\alpha(-x, z, y) = W_\alpha(-z, y, -x).$$

In [4], it has been shown the following inequality

$$\int_{\mathbb{R}} |W_\alpha(x, y, z)|d\mu_\alpha(z) \leq \sqrt{2}.$$

The Dunkl kernel  $E_\alpha$  satisfies the following product formula (see [15])

$$E_\alpha(ixt)E_\alpha(iyt) = \int_{\mathbb{R}} E_\alpha(itz)d\gamma_{x,y}(z), \quad x, y, t \in \mathbb{R},$$

where  $\gamma_{x,y}$  is a signed measure on  $\mathbb{R}$  given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x, y, z)d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R} \setminus \{0\} \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \tag{2.2}$$

with  $\text{supp}\gamma_{x,y} = S_{x,y} \cup (-S_{x,y})$ .

For  $x, y \in \mathbb{R}$  and  $f$  a continuous function on  $\mathbb{R}$ , the Dunkl translation operator  $\tau_x$  is given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z)$$

and satisfies the following properties :

- $\tau_x$  is a continuous linear operator from  $\mathcal{E}(\mathbb{R})$  into itself.
- For all  $f \in \mathcal{E}(\mathbb{R})$ , we have

$$\tau_x(f)(y) = \tau_y(f)(x) \quad \text{and} \quad \tau_0(f)(x) = f(x) \tag{2.3}$$

$$\tau_x \circ \tau_y = \tau_y \circ \tau_x \quad \text{and} \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha. \tag{2.4}$$

- For all  $x \in \mathbb{R}$ , the operator  $\tau_x$  extends to  $L^p(\mu_\alpha)$ ,  $p \geq 1$  and we have for  $f \in L^p(\mu_\alpha)$

$$\|\tau_x(f)\|_{p,\alpha} \leq \sqrt{2}\|f\|_{p,\alpha}. \tag{2.5}$$

The Dunkl convolution  $f *_\alpha g$  of two continuous functions  $f$  and  $g$  on  $\mathbb{R}$  with compact support, is defined by

$$(f *_\alpha g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y)g(y)d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The convolution  $*_\alpha$  is associative and commutative and satisfies the following properties:

- Assume that  $p, q, r \in [1, +\infty[$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$  (the Young condition). Then the map  $(f, g) \rightarrow f *_\alpha g$  defined on  $C_c(\mathbb{R}) \times C_c(\mathbb{R})$ , extends to a continuous map from  $L^p(\mu_\alpha) \times L^q(\mu_\alpha)$  to  $L^r(\mu_\alpha)$  and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq \sqrt{2}\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \tag{2.6}$$

- For all  $f \in L^1(\mu_\alpha)$ ,  $g \in L^2(\mu_\alpha)$  and  $h \in L^p(\mu_\alpha)$ ,  $1 \leq p < +\infty$ , we have

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g),$$

$$\text{and} \quad \tau_t(f *_\alpha h) = \tau_t(f) *_\alpha h = f *_\alpha \tau_t(h), \quad t \in \mathbb{R}. \tag{2.7}$$

It has been shown in [12], the following generalized Taylor formula with integral remainder:

**Proposition 2.1.** *For  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $a \in \mathbb{R}$ , we have*

$$\tau_x f(a) = \sum_{p=0}^{k-1} b_p(x) \Lambda_\alpha^p f(a) + R_k(x, f)(a), \quad x \in \mathbb{R} \setminus \{0\}, \tag{2.8}$$

with  $R_k(x, f)(a)$  is the integral remainder of order  $k$  given by

$$R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_{k-1}(x, y) \tau_y(\Lambda_\alpha^k f)(a) A_\alpha(y) dy, \tag{2.9}$$

where

$$\begin{aligned}
 \text{i) } & b_{2m}(x) = \frac{1}{(\alpha+1)_m m!} \left(\frac{x}{2}\right)^{2m}, \quad b_{2m+1}(x) = \frac{1}{(\alpha+1)_{m+1} m!} \left(\frac{x}{2}\right)^{2m+1}, \text{ for} \\
 & \text{all } m \in \mathbb{N}. \\
 \text{ii) } & \Theta_{k-1}(x, y) = u_{k-1}(x, y) + v_{k-1}(x, y) \text{ with } u_0(x, y) = \frac{\text{sgn}(x)}{2A_\alpha(x)}, \\
 & v_0(x, y) = \frac{\text{sgn}(y)}{2A_\alpha(y)}, \quad u_k(x, y) = \int_{|y|}^{|x|} v_{k-1}(x, z) dz \text{ and} \\
 & v_k(x, y) = \frac{\text{sgn}(y)}{A_\alpha(y)} \int_{|y|}^{|x|} u_{k-1}(x, z) A_\alpha(z) dz.
 \end{aligned}$$

According to ([17], Lemma 2.2), the Dunkl operator  $\Lambda_\alpha$  have the following regularity properties:

$$\Lambda_\alpha \text{ leaves } \mathcal{C}_c^\infty(\mathbb{R}) \text{ and the Schwartz space } \mathcal{S}(\mathbb{R}) \text{ invariant.} \quad (2.10)$$

### 3. SOME PROPERTIES OF THE INTEGRAL REMAINDER OF ORDER $k$

In this section, we prove some properties and estimates of the integral remainder in the generalized Taylor formula.

*Remark 3.1.* Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $x \in \mathbb{R} \setminus \{0\}$ .

1/ From Proposition 2.1, we have

$$\begin{aligned}
 R_k(x, f) &= \tau_x(f) - f - b_1(x)\Lambda_\alpha f \dots - b_{k-1}(x)\Lambda_\alpha^{k-1} f \\
 &= R_{k-1}(x, f) - b_{k-1}(x)\Lambda_\alpha^{k-1} f,
 \end{aligned} \quad (3.1)$$

where we put for  $k = 1$ ,  $R_0(x, f) = \tau_x(f)$ . Observe that

$$R_1(x, f) = R_0(x, f) - b_0(x)\Lambda_\alpha^0 f = \tau_x(f) - f.$$

2/ According to ([12], p.352) and Proposition 2.1, i), we have

$$\begin{aligned}
 \int_{-|x|}^{|x|} |\Theta_{k-1}(x, y)| A_\alpha(y) dy &\leq b_k(|x|) + |x| b_{k-1}(|x|) \\
 &\leq c|x|^k.
 \end{aligned} \quad (3.2)$$

3/ Note that the function  $y \mapsto \tau_y(f) - f$  is continuous on  $\mathbb{R}$  (see [11], Lemma 1, (ii)), which implies that the same is true for the function  $y \mapsto R_k(y, f)$ .

**Lemma 3.2.** *Let  $k = 1, 2, \dots$ , then for all  $f \in \mathcal{E}(\mathbb{R})$  satisfying  $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$ , we have*

$$\|R_{k-1}(x, f)\|_{p, \alpha} \leq c|x|^{k-1} \|\Lambda_\alpha^{k-1} f\|_{p, \alpha}, \quad x \in \mathbb{R} \setminus \{0\}. \quad (3.3)$$

*Proof.* Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$  and  $x \in \mathbb{R} \setminus \{0\}$ . For  $k = 1$ , by (2.5), it's clear that  $\|R_0(x, f)\|_{p, \alpha} = \|\tau_x(f)\|_{p, \alpha} \leq c\|f\|_{p, \alpha}$ . Using the Minkowski's inequality for integrals, (2.5) and (2.9), we have for  $k \geq 2$

$$\begin{aligned}
 \|R_{k-1}(x, f)\|_{p, \alpha} &\leq \int_{-|x|}^{|x|} |\Theta_{k-2}(x, y)| \|\tau_y(\Lambda_\alpha^{k-1} f)\|_{p, \alpha} A_\alpha(y) dy \\
 &\leq c \|\Lambda_\alpha^{k-1} f\|_{p, \alpha} \int_{-|x|}^{|x|} |\Theta_{k-2}(x, y)| A_\alpha(y) dy.
 \end{aligned}$$

From (3.2), we deduce our result. □

*Remark 3.3.* Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $x \in \mathbb{R} \setminus \{0\}$ .

1/ If  $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$ , then we have by (3.1), (3.3) and Proposition 2.1, i),

$$\begin{aligned} \|R_k(x, f)\|_{p,\alpha} &\leq \|R_{k-1}(x, f)\|_{p,\alpha} + \|b_{k-1}(x)\Lambda_\alpha^{k-1} f\|_{p,\alpha} \\ &\leq c|x|^{k-1}\|\Lambda_\alpha^{k-1} f\|_{p,\alpha}. \end{aligned} \tag{3.4}$$

2/ We observe from Proposition 2.1 that

$$\begin{aligned} R_k(x, f) + R_k(-x, f) &= \tau_x(f) + \tau_{-x}(f) - \sum_{p=0}^{k-1} (b_p(x) + b_p(-x))\Lambda_\alpha^p f \\ &= \tau_x(f) + \tau_{-x}(f) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x)\Lambda_\alpha^{2i} f. \end{aligned} \tag{3.5}$$

#### 4. CHARACTERIZATIONS OF BESOV–DUNKL SPACES OF ORDER $k$

In this section, we begin with a remark, a proposition containing sufficient conditions and an example.

*Remark 4.1.* Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1} f$  is in  $L^p(\mu_\alpha)$  and  $x \in (0, +\infty)$ .

1/ We can assert from (3.1) that

$$\omega_{p,\alpha}^k(x, f) = \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha}, \text{ for } k > 1. \tag{4.1}$$

2/ For  $k = 1$ ,  $\omega_{p,\alpha}^k(x, f) = \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}$ , called the modulus of continuity of second order of  $f$ . In this case, we recover with this expression the Besov–Dunkl spaces defined in [1, 5, 9].

**Proposition 4.2.** *Let  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$ ,  $0 < \beta < 1$  and  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$  for  $k = 1, 2, \dots$ . If  $\Lambda_\alpha^k f$  are in  $L^p(\mu_\alpha)$ , then  $f \in \mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ .*

*Proof.* Let  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$ ,  $0 < \beta < 1$  and  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1} f, \Lambda_\alpha^k f$  are in  $L^p(\mu_\alpha)$  for  $k = 1, 2, \dots$ . By (3.3), (3.4) and (4.1), we obtain for  $x \in (0, +\infty)$

$$\omega_{p,\alpha}^k(x, f) \leq c x^k \|\Lambda_\alpha^k f\|_{p,\alpha} \quad \text{and} \quad \omega_{p,\alpha}^k(x, f) \leq c x^{k-1} \|\Lambda_\alpha^{k-1} f\|_{p,\alpha}.$$

Then we can write,

$$\int_0^{+\infty} \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} \leq c \int_0^1 \left( \frac{\|\Lambda_\alpha^k f\|_{p,\alpha}}{x^{\beta-1}} \right)^q \frac{dx}{x} + c \int_1^{+\infty} \left( \frac{\|\Lambda_\alpha^{k-1} f\|_{p,\alpha}}{x^\beta} \right)^q \frac{dx}{x},$$

giving two finite integrals. Here when  $q = +\infty$ , we make the usual modification. □

**Example 4.3.** From (2.10) and Proposition 4.2, we can assert that the spaces  $\mathcal{C}_c^\infty(\mathbb{R})$  and  $\mathcal{S}(\mathbb{R})$  are included in  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \cap L^p(\mu_\alpha)$ .

In order to establish that  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \cap L^p(\mu_\alpha) = \mathcal{C}_{p,q,\phi}^{k,\beta,\alpha}$ , we give an example of functions in the class  $\mathcal{A}_k$  and we prove some useful lemmas.

**Example 4.4.** According to ([16], Example 3.3,(2)), the generalized Hermite polynomials on  $\mathbb{R}$ , denoted by  $H_n^{\alpha+\frac{1}{2}}$ ,  $n \in \mathbb{N}$  are orthogonal with respect to the measure  $e^{-x^2}d\mu_\alpha(x)$  and can be written as

$$H_{2n}^{\alpha+\frac{1}{2}}(x) = (-1)^n 2^{2n} n! L_n^\alpha(x^2) \quad \text{and} \quad H_{2n+1}^{\alpha+\frac{1}{2}}(x) = (-1)^n 2^{2n+1} n! x L_n^{\alpha+1}(x^2),$$

where the  $L_n^\alpha$  are the Laguerre polynomials of index  $\alpha \geq -\frac{1}{2}$ , given by

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}).$$

For  $k = 1, 2, \dots$ , fix any positive integer  $n_0 > [\frac{k-1}{2}]$  and take for example the function defined on  $\mathbb{R}$  by  $\phi(x) = H_{2n_0}^{\alpha+\frac{1}{2}}(x) e^{-x^2}$ . Put  $P_i(x) = x^{2i}$  for  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$ , since  $P_i \in \text{span}_{\mathbb{R}} \{H_p^{\alpha+\frac{1}{2}}, p = 0, 1, \dots, 2[\frac{k-1}{2}]\}$ , then we can assert that  $\phi \in \mathcal{S}_*(\mathbb{R})$  and satisfy  $\int_0^{+\infty} x^{2i} \phi(x) d\mu_\alpha(x) = 0$ , which gives that  $\phi \in \mathcal{A}_k$ .

**Lemma 4.5.** Let  $k = 1, 2, \dots$ ,  $\phi \in \mathcal{A}_k$ ,  $1 \leq p < +\infty$  and  $r > 0$ , then for all  $f \in \mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$  satisfying  $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$  and  $t > 0$ , we have

$$\|\phi_t *_\alpha f\|_{p,\alpha} \leq c \int_0^{+\infty} \min \left\{ \left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r \right\} \omega_{p,\alpha}^k(x, f) \frac{dx}{x}. \tag{4.2}$$

*Proof.* Let  $k = 1, 2, \dots$ ,  $t > 0$ , we have for  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$ ,

$$\int_0^{+\infty} x^{2i} \phi(x) d\mu_\alpha(x) = 0 \implies \int_0^{+\infty} x^{2i} \phi_t(x) d\mu_\alpha(x) = 0, \tag{4.3}$$

where  $\phi_t$  is the dilatation of  $\phi$ .

We observe that,

$$\begin{aligned} (\phi_t *_\alpha f)(y) &= \int_{\mathbb{R}} \phi_t(x) \tau_y(f)(-x) d\mu_\alpha(x) \\ &= \int_{\mathbb{R}} \phi_t(x) \tau_y(f)(x) d\mu_\alpha(x), \end{aligned}$$

then using (2.3), (3.5), (4.3) and Proposition 2.1, we can write for  $y \in \mathbb{R}$

$$\begin{aligned} &2(\phi_t *_\alpha f)(y) \\ &= \int_{\mathbb{R}} \phi_t(x) \left( \tau_y(f)(x) + \tau_y(f)(-x) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_\alpha^{2i} f(y) \right) d\mu_\alpha(x) \\ &= 2 \int_0^{+\infty} \phi_t(x) \left( \tau_x(f)(y) + \tau_{-x}(f)(y) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_\alpha^{2i} f(y) \right) d\mu_\alpha(x) \\ &= 2 \int_0^{+\infty} \phi_t(x) (R_k(x, f)(y) + R_k(-x, f)(y)) d\mu_\alpha(x). \end{aligned}$$



By Minkowski's inequality for integrals, we obtain

$$\begin{aligned} \|\phi_t *_{\alpha} f\|_{p,\alpha} &\leq \int_0^{+\infty} |\phi_t(x)| \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} d\mu_{\alpha}(x) \\ &\leq c \int_0^{+\infty} \left(\frac{x}{t}\right)^{2(\alpha+1)} \left|\phi\left(\frac{x}{t}\right)\right| \omega_{p,\alpha}^k(x, f) \frac{dx}{x} \end{aligned} \quad (4.4)$$

$$\leq c \int_0^{+\infty} \left(\frac{x}{t}\right)^{2(\alpha+1)} \omega_{p,\alpha}^k(x, f) \frac{dx}{x}. \quad (4.5)$$

On the other hand, since  $\phi \in \mathcal{S}_*(\mathbb{R})$ , then from (4.4) and for  $r > 0$  there exists a constant  $c$  such that

$$\|\phi_t *_{\alpha} f\|_{p,\alpha} \leq c \int_0^{+\infty} \left(\frac{t}{x}\right)^r \omega_{p,\alpha}^k(x, f) \frac{dx}{x}. \quad (4.6)$$

Using (4.5) and (4.6), we deduce our result.  $\square$

**Lemma 4.6.** *Let  $k = 1, 2, \dots$ ,  $1 < p < +\infty$  and  $\phi \in \mathcal{A}_k$ , then for all  $f \in \mathcal{E}(\mathbb{R})$  satisfying  $\Lambda_{\alpha}^{2i} f \in L^p(\mu_{\alpha})$ ,  $0 \leq i \leq [\frac{k-1}{2}]$  and  $x > 0$ , we have*

$$\omega_{p,\alpha}^k(x, f) \leq c \int_0^{+\infty} \min \left\{ \left(\frac{x}{t}\right)^{k-1}, \left(\frac{x}{t}\right)^k \right\} \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t}. \quad (4.7)$$

*Proof.* Put for  $0 < \varepsilon < \delta < +\infty$

$$f_{\varepsilon,\delta}(y) = \int_{\varepsilon}^{\delta} (\phi_t *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t}, \quad y \in \mathbb{R}.$$

Then for  $i \in \mathbb{N}$ , we have

$$(\Lambda_{\alpha}^{2i} f)_{\varepsilon,\delta}(y) = \int_{\varepsilon}^{\delta} (\Lambda_{\alpha}^{2i} \phi_t *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t}, \quad y \in \mathbb{R}.$$

From the integral representation of  $\tau_x$ , we obtain by interchanging the orders of integration and (2.7),

$$\begin{aligned} \tau_x(f_{\varepsilon,\delta})(y) &= \int_{\varepsilon}^{\delta} \tau_x(\phi_t *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t} \\ &= \int_{\varepsilon}^{\delta} (\tau_x(\phi_t) *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t}, \quad y \in \mathbb{R}, \quad x \in (0, +\infty), \end{aligned}$$

so we can write for  $x \in (0, +\infty)$  and  $y \in \mathbb{R}$ ,

$$(R_k(x, f_{\varepsilon,\delta}) + R_k(-x, f_{\varepsilon,\delta}))(y)$$

$$= \int_{\varepsilon}^{\delta} \left[ (\tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t) *_{\alpha} \phi_t *_{\alpha} f \right](y) \frac{dt}{t}.$$

Using Minkowski's inequality for integrals and (2.6), we get

$$\begin{aligned}
 & \| (R_k(x, f_{\varepsilon, \delta}) + R_k(-x, f_{\varepsilon, \delta})) \|_{p, \alpha} \\
 & \leq \int_{\varepsilon}^{\delta} \| (\tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t) *_{\alpha} \phi_t *_{\alpha} f \|_{p, \alpha} \frac{dt}{t} \\
 & \leq c \int_{\varepsilon}^{\delta} \| (\tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t) \|_{1, \alpha} \| \phi_t *_{\alpha} f \|_{p, \alpha} \frac{dt}{t} \\
 & = c \int_{\varepsilon}^{\delta} \| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} \| \phi_t *_{\alpha} f \|_{p, \alpha} \frac{dt}{t}. \tag{4.8}
 \end{aligned}$$

For  $x, t \in (0, +\infty)$ , we have

$$\begin{aligned}
 & \| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} \\
 & = \| \tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t \|_{1, \alpha} \\
 & = \int_{\mathbb{R}} \left| \left( \int_{\mathbb{R}} \phi_t(z) (d\gamma_{x,y}(z) + d\gamma_{-x,y}(z)) \right) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t(y) \right| d\mu_{\alpha}(y) \\
 & = \int_{\mathbb{R}} \left| \left( \int_{\mathbb{R}} \phi\left(\frac{z}{t}\right) (d\gamma_{x,y}(z) + d\gamma_{-x,y}(z)) \right) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi\left(\frac{y}{t}\right) \right| \frac{1}{t^{2(\alpha+1)}} d\mu_{\alpha}(y).
 \end{aligned}$$

By (2.2) and the change of variable  $z' = \frac{z}{t}$ , we have

$$W_{\alpha}(x, y, z't) t^{2(\alpha+1)} = W_{\alpha}\left(\frac{x}{t}, \frac{y}{t}, z'\right),$$

which implies that  $d\gamma_{x,y}(z) = d\gamma_{\frac{x}{t}, \frac{y}{t}}(z')$ . Hence, we obtain

$$\begin{aligned}
 & \| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} \\
 & = \int_{\mathbb{R}} \left| \left( \int_{\mathbb{R}} \phi(z') (d\gamma_{\frac{x}{t}, \frac{y}{t}}(z') + d\gamma_{\frac{-x}{t}, \frac{y}{t}}(z')) \right) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi\left(\frac{y}{t}\right) \right| \frac{1}{t^{2(\alpha+1)}} d\mu_{\alpha}(y) \\
 & = \int_{\mathbb{R}} \left| \left( \tau_{\frac{x}{t}}(\phi)\left(\frac{y}{t}\right) + \tau_{\frac{-x}{t}}(\phi)\left(\frac{y}{t}\right) \right) \frac{1}{t^{2(\alpha+1)}} - 2 \left( \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi \right)_t(y) \right| d\mu_{\alpha}(y) \\
 & = \left\| \left( \tau_{\frac{x}{t}}(\phi) + \tau_{\frac{-x}{t}}(\phi) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi \right)_t \right\|_{1, \alpha} \\
 & = \left\| \tau_{\frac{x}{t}}(\phi) + \tau_{\frac{-x}{t}}(\phi) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi \right\|_{1, \alpha},
 \end{aligned}$$

which gives

$$\| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} = \left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1, \alpha}. \tag{4.9}$$

Since  $\phi \in \mathcal{S}_*(\mathbb{R})$ , then using (2.10) and (3.3), we can assert that

$$\left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^k \|\Lambda_\alpha^k \phi\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^k,$$

on the other hand, by (3.4) we have

$$\left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^{k-1} \|\Lambda_\alpha^{k-1} \phi\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^{k-1},$$

then we get,

$$\left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1,\alpha} \leq c \min \left\{ \left(\frac{x}{t}\right)^{k-1}, \left(\frac{x}{t}\right)^k \right\}. \quad (4.10)$$

From (3.6), (4.8), (4.9) and (4.10), we obtain

$$\omega_{p,\alpha}^k(x, f_{\varepsilon,\delta}) \leq c \int_\varepsilon^\delta \min \left\{ \left(\frac{x}{t}\right)^{k-1}, \left(\frac{x}{t}\right)^k \right\} \|\phi_t *_\alpha f\|_{p,\alpha} \frac{dt}{t}. \quad (4.11)$$

Note that  $\Lambda_\alpha^{2i} \phi *_\alpha \phi \in \mathcal{S}_*(\mathbb{R})$ . By (2.1) and (2.7), we have

$$\begin{aligned} \int_{\mathbb{R}} (\Lambda_\alpha^{2i} \phi *_\alpha \phi)(x) |x|^{2\alpha+1} dx &= 2^{\alpha+1} \Gamma(\alpha+1) \mathcal{F}_\alpha(\Lambda_\alpha^{2i} \phi *_\alpha \phi)(0) \\ &= 2^{\alpha+1} \Gamma(\alpha+1) \mathcal{F}_\alpha(\Lambda_\alpha^{2i} \phi)(0) \mathcal{F}_\alpha(\phi)(0) \\ &= 2^{\alpha+1} \Gamma(\alpha+1) \mathcal{F}_\alpha(\Lambda_\alpha^{2i} \phi)(0) \int_{\mathbb{R}} \phi(z) d\mu_\alpha(z) = 0. \end{aligned}$$

Since  $\Lambda_\alpha^{2i} \phi *_\alpha \phi$  is in the Schwartz space  $\mathcal{S}(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} |\log|x|| |\Lambda_\alpha^{2i} \phi *_\alpha \phi(x)| |x|^{2\alpha+1} dx < +\infty.$$

Then, by Calderón's reproducing formula related to the Dunkl operator (see [11], Theorem 3), we have

$$\lim_{\varepsilon \rightarrow 0, \delta \rightarrow +\infty} (\Lambda_\alpha^{2i} f)_{\varepsilon,\delta} = c \Lambda_\alpha^{2i} f, \quad \text{in } L^p(\mu_\alpha),$$

hence from (4.11), we deduce our result.  $\square$

**Theorem 4.7.** *Let  $0 < \beta < 1$ ,  $k = 1, 2, \dots$ ,  $1 < p < +\infty$  and  $1 \leq q \leq +\infty$ , then we have*

$$\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \cap L^p(\mu_\alpha) = \mathcal{C}_{p,q,\phi}^{k,\beta,\alpha},$$

and for  $p = 1$ , we have only  $\mathcal{B}^k \mathcal{D}_{1,q}^{\beta,\alpha} \cap L^p(\mu_\alpha) \subset \mathcal{C}_{1,q,\phi}^{k,\beta,\alpha}$ .

*Proof.* Assume  $f \in \mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \cap L^p(\mu_\alpha)$  for  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$  and  $r > \beta + k - 1$ .

• Case  $q = 1$ . By (4.2) and Fubini's theorem, we have

$$\begin{aligned}
 & \int_0^{+\infty} \frac{\|f *_{\alpha} \phi_t\|_{p,\alpha}}{t^{\beta+k-1}} \frac{dt}{t} \\
 & \leq c \int_0^{+\infty} \int_0^{+\infty} \min \left\{ \left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r \right\} \omega_{p,\alpha}^k(x, f) t^{-\beta-k} dt \frac{dx}{x} \\
 & \leq c \int_0^{+\infty} \omega_{p,\alpha}^k(x, f) \left( \int_0^{+\infty} \min \left\{ \left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r \right\} t^{-\beta-k} dt \right) \frac{dx}{x} \\
 & \leq c \int_0^{+\infty} \omega_{p,\alpha}^k(x, f) \left( x^{-r} \int_0^x t^{r-\beta-k} dt + x^{2(\alpha+1)} \int_x^{+\infty} t^{-\beta-k-2\alpha-2} dt \right) \frac{dx}{x} \\
 & \leq c \int_0^{+\infty} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \frac{dx}{x} < +\infty,
 \end{aligned}$$

hence  $f \in \mathcal{C}_{p,1,\phi}^{k,\beta,\alpha}$ .

• Case  $q = +\infty$ . By (4.2), we have

$$\begin{aligned}
 & \|\phi_t *_{\alpha} f\|_{p,\alpha} \\
 & \leq c \left( \int_0^t \left(\frac{x}{t}\right)^{2(\alpha+1)} \omega_{p,\alpha}^k(x, f) \frac{dx}{x} + \int_t^{+\infty} \left(\frac{t}{x}\right)^r \omega_{p,\alpha}^k(x, f) \frac{dx}{x} \right) \\
 & \leq c \sup_{x \in (0, +\infty)} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \left( t^{-2(\alpha+1)} \int_0^t x^{2\alpha+\beta+k} dx + t^r \int_t^{+\infty} x^{\beta+k-r-2} dx \right) \\
 & \leq c t^{\beta+k-1} \sup_{x \in (0, +\infty)} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}},
 \end{aligned}$$

then we deduce that  $f \in \mathcal{C}_{p,\infty,\phi}^{k,\beta,\alpha}$ .

• Case  $1 < q < +\infty$ . By (4.2) again, we have for  $t > 0$

$$\frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \leq c \int_0^{+\infty} \left(\frac{x}{t}\right)^{\beta+k-1} \min \left\{ \left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r \right\} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \frac{dx}{x}.$$

Put  $L(x, t) = \left(\frac{x}{t}\right)^{\beta+k-1} \min \left\{ \left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r \right\}$  and  $q' = \frac{q}{q-1}$  the conjugate of  $q$ . Since

$$\int_0^{+\infty} L(x, t) \frac{dx}{x} = t^{-\beta-k-2\alpha-1} \int_0^t x^{\beta+k+2\alpha} dx + t^{-\beta-k+r+1} \int_t^{+\infty} x^{\beta+k-r-2} dx \leq c,$$

we can write using Hölder's inequality,

$$\begin{aligned}
 \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} & \leq c \int_0^{+\infty} (L(x, t))^{\frac{1}{q'}} \left( (L(x, t))^{\frac{1}{q}} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right) \frac{dx}{x} \\
 & \leq c \left( \int_0^{+\infty} L(x, t) \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} \right)^{\frac{1}{q}}.
 \end{aligned}$$

By the fact that

$$\int_0^{+\infty} L(x, t) \frac{dt}{t} = x^{\beta+k-r-1} \int_0^x t^{-\beta-k+r} dt + x^{\beta+k+2\alpha+1} \int_x^{+\infty} t^{-\beta-k-2\alpha-2} dt \leq c,$$

we get by using Fubini's theorem,

$$\begin{aligned} \int_0^{+\infty} \left( \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} &\leq c \int_0^{+\infty} \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \left( \int_0^{+\infty} L(x, t) \frac{dt}{t} \right) \frac{dx}{x} \\ &\leq c \int_0^{+\infty} \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} < +\infty, \end{aligned}$$

which proves the result.

Assume now  $f \in \mathcal{C}_{p,q,\phi}^{k,\beta,\alpha}$  for  $1 < p < +\infty$  and  $1 \leq q \leq +\infty$ .

- Case  $q = 1$ . By (4.7) and Fubini's theorem, we have

$$\begin{aligned} \int_0^{+\infty} \frac{\omega_p^\alpha(f)(x)}{x^{\beta+k-1}} \frac{dx}{x} &\leq c \int_0^{+\infty} \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} \|\phi_t *_{\alpha} f\|_{p,\alpha} x^{-\beta-k} \frac{dt}{t} dx \\ &\leq c \int_0^{+\infty} \|\phi_t *_{\alpha} f\|_{p,\alpha} \left( \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} x^{-\beta-k} dx \right) \frac{dt}{t} \\ &\leq c \int_0^{+\infty} \|\phi_t *_{\alpha} f\|_{p,\alpha} \left( \frac{1}{t^k} \int_0^t x^{-\beta} dx + \frac{1}{t^{k-1}} \int_t^{+\infty} x^{-\beta-1} dx \right) \frac{dt}{t} \\ &\leq c \int_0^{+\infty} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \frac{dt}{t} < +\infty, \end{aligned}$$

then we obtain the result.

- Case  $q = +\infty$ . By (4.7), we get

$$\begin{aligned} \omega_p^\alpha(f)(x) &\leq c \left( \int_0^x \left( \frac{x}{t} \right)^{k-1} \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t} + \int_x^{+\infty} \left( \frac{x}{t} \right)^k \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t} \right) \\ &\leq c \sup_{t \in (0, +\infty)} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \left( x^{k-1} \int_0^x t^{\beta-1} dt + x^k \int_x^{+\infty} t^{\beta-2} dt \right) \\ &\leq c x^{\beta+k-1} \sup_{t \in (0, +\infty)} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}}, \end{aligned}$$

so, we deduce that  $f \in \mathcal{B}^k \mathcal{D}_{p,\infty}^{\beta,\alpha} \cap L^p(\mu_\alpha)$ .

- Case  $1 < q < +\infty$ . By (4.7) again, we have for  $x > 0$

$$\frac{\omega_p^\alpha(f)(x)}{x^{\beta+k-1}} \leq c \int_0^{+\infty} \left( \frac{t}{x} \right)^{\beta+k-1} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \frac{dt}{t}.$$

Note that

$$\left( \frac{t}{x} \right)^{\beta+k-1} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} = \left( \frac{t}{x} \right)^\beta \min \left\{ 1, \frac{x}{t} \right\}.$$

Put  $G(x, t) = \left( \frac{t}{x} \right)^\beta \min \left\{ 1, \frac{x}{t} \right\}$  and  $q'$  the conjugate of  $q$ . Since

$$\int_0^{+\infty} G(x, t) \frac{dt}{t} = x^{-\beta} \int_0^x t^{\beta-1} dt + x^{-\beta+1} \int_x^{+\infty} t^{\beta-2} dt \leq c,$$

then using Hölder's inequality, we can write

$$\begin{aligned} \frac{\omega_p^\alpha(f)(x)}{x^{\beta+k-1}} &\leq c \int_0^{+\infty} (G(x,t))^{\frac{1}{q'}} \left( (G(x,t))^{\frac{1}{q}} \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right) \frac{dt}{t} \\ &\leq c \left( \int_0^{+\infty} G(x,t) \left( \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

By the fact that

$$\int_0^{+\infty} G(x,t) \frac{dx}{x} = t^{\beta-1} \int_0^t x^{-\beta} dx + t^\beta \int_t^{+\infty} x^{-\beta-1} dx \leq c,$$

we get by using Fubini's theorem,

$$\begin{aligned} \int_0^{+\infty} \left( \frac{\omega_p^\alpha(f)(x)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} &\leq c \int_0^{+\infty} \left( \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \left( \int_0^{+\infty} G(x,t) \frac{dx}{x} \right) \frac{dt}{t} \\ &\leq c \int_0^{+\infty} \left( \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} < +\infty, \end{aligned}$$

thus the result is established.  $\square$

*Remark 4.8.* From theorem 4.7, we can assert that  $\mathcal{C}_{p,q,\phi}^{k,\beta,\alpha}$  is independent of the specific selection of the function  $\phi$  in  $\mathcal{A}_k$ .

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## REFERENCES

1. C. Abdelkefi and M. Sifi, *Characterisation of Besov spaces for the Dunkl operator on the real line*, J. Inequal. Pure Appl. Math. **8** (2007), Issue 3, Article 73, 11 pp.
2. C. Abdelkefi, J. Ph. Anker, F. Sassi, and M. Sifi, *Besov-type spaces on  $\mathbb{R}^d$  and integrability for the Dunkl transform*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 019, 15 pp.
3. C. Abdelkefi, *Weighted function spaces and Dunkl transform*, Mediterr. J. Math. **9** (2012), no. 3, 499–513 Springer.
4. B. Amri, J. Ph. Anker, and M. Sifi, *Three results in Dunkl analysis*, Colloq. Math. **118** (2010), no. 1, 299–312.
5. J. L. Ansorena and O. Blasco, *Characterization of weighted Besov spaces*, Math. Nachr. **171** (1995), 5–17.
6. O. V. Besov, *On a family of function spaces in connection with embeddings and extention theorems*, (Russian) Trudy. Mat. Inst. Steklov. **60** (1961), 42–81.
7. C. F. Dunkl, *Differential-difference operators associated to reflexion groups*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 167–183.
8. D. V. Giang and F. Moricz, *A new characterization of Besov spaces on the real line*, J. Math. Anal. Appl. **189** (1995), no. 2, 533–551.
9. L. Kamoun, *Besov-type spaces for the Dunkl operators on the real line*, J. Comput. Appl. Math. **199**, no. 1, (2007), 299–312.
10. J. Löfström and J. Peetre, *Approximation theorems connected with generalized translations*, Math. Ann. **181** (1969), 255–268.
11. M. A. Mourou and K. Trimèche, *Calderon's reproducing formula related to the Dunkl operator on the real line*, Monatsh. Math. **136** (2002), no. 1, 47–65.

12. M. A. Mourou, *Taylor series associated with a differential-difference operator on the real line*, Proceedings of the Sixth International Symposium on Orthogonal Polynomials, Special Functions and their Applications (Rome, 2001). J. Comp. and Appl. Math., **153** (2003), 343–354.
13. J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series, Durham, NC, 1976.
14. M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus*, Non-selfadjoint operators and related topics (Beer Sheva, 1992), 369–396, Oper. Theory Adv. Appl., 73, Birkhäuser, Basel 1994.
15. M. Rösler, *Bessel-Type signed hypergroup on  $\mathbb{R}$* , Probability measures on groups and related structures, XI (Oberwolfach, 1994), 292–304, World Sci. Publ., River Edge, NJ, 1995.
16. M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. **192** (1998), no. 3, 519–541.
17. M. Rösler, *Dunkl operators: theory and applications* Orthogonal polynomials and special functions (Leuven, 2002), 93–135, Lecture Notes in Math., 1817, Springer, Berlin, 2003.

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