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ON DIFFERENT TYPE OF FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS VIA MEASURE OF NONCOMPACTNESS

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Abstract. In this paper by using the measure of noncompactness concept, we present new fixed point theorems for multivalued maps. In further we introduce a new class of mappings which are general than Meir–Keeler mappings. Finally, we use these results to investigate the existence of weak solutions to an Evolution differential inclusion with lack of compactness.

1. Introduction

Recently many papers have appeared about generalizations of Darbo's fixed point and its applications. For example, in 2015 Aghajani and Mursaleen [12] introduced the definition of a Meir Keeler condensing operator and proved a theorem that guarantees the existence of fixed points for single valued mappings and proved a fixed point theorem which extended the well-known Darbo's and Meir Keeler fixed point theorems. Another generalization is due to Samadia and Ghaemia [14], where they proved the existence of fixed points under a more general condition than the contraction condition. It is interesting to see what happened in the multivalued case and whether these results still hold.

Our aim in this paper is to recall some essential concepts and results that are needed throughout this work. Then, we give a version of a Meir Keeler theorem for condensing multivalued mappings, we also present some related results and applications. In the third section, we present a version of theorems presented

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in [14] for multivalued mappings and some related results. We also study the existence of fixed point for multivalued power set contraction mappings.

Finally, in order to indicate their applicability, we choose one among the previous theorems and we use it to study the existence of mild solutions for a nonlocal differential evolution inclusion.

2. Preliminaries

In this section, we survey some definitions and preliminary facts for measure of noncompactness and multivalued analysis which will be used in this paper.

Let (X, d) and (Y, d') be two metric spaces. We use the following notations,

$$\mathcal{P}_{cl}(X) = \{A \in \mathcal{P}(X) : A \textit{ closed}\}, \quad \mathcal{P}_b(X) = \{A \in \mathcal{P}(X) : A \textit{ bounded}\},$$

$$\mathcal{P}_{cv}(A) = \{A \in \mathcal{P}(X) : A \textit{ convex}\}, \quad \mathcal{P}_{cp}(X) = \{A \in \mathcal{P}(X) : Y \textit{ compact}\}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}) \ , \ \sup_{b \in \mathcal{B}} d(\mathcal{A}, b) \right\},$$

where $d(\mathcal{A}, b) = \inf_{a \in \mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b \in \mathcal{B}} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [11]).

We denote by

$$S_F(y) = \{f \in L^1(J, X) : f(t) \in F(t, y_{\rho(t, y_t)}) \ , \ a.e. \ t \in J\},$$

the set of *selectors* of F.

$C(E; X)$ is the Banach space of all continuous mappings from E into X with the norm

$$\|y\| = \sup \{ \ |y(t)| : t \in H \ \}.$$

$B(X)$ is the space of all bounded linear mappings F from X into X, with the norm

$$\|F\|_{B(X)} = \sup \{ \ |F(y)| \ : \ |y| = 1 \ \}.$$

A multivalued map $F : X \to \mathcal{P}(X)$ has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

A multivalued map $F : X \to \mathcal{P}(X)$ is said to be convex (closed) valued if $F(x)$ is convex (closed) in Y for each set A of X and F is bounded valued if $F(x)$ is bounded in Y for each $x \in X$, i,e

$$\sup_{x \in A} \{\sup\{ \ \|y\| \ : \ y \in F(x)\}\} < \infty.$$

In further, F is compact if $F(A)$ is relatively compact for every $B \in \mathcal{P}_b(X)$. Finally, F is upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open subset U of X containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subseteq U$.

Lemma 2.1. *Assume that $D \subset X$ and Fx is closed for all $x \in D$, then the following conclusions hold,*

i) *if F is u.s.c. and D is closed, then F has a closed graph (i.e., $x_n \to x$ and $y_n \to y$ such that $y_n \in F(x_n) \Rightarrow y \in F(x)$).*

ii) *if $F(D)$ is compact and D is closed, then F is u.s.c. if and only if F has a closed graph.*

For more details on multivalued maps we refer to the books of Deimling [5], Górniewicz [9].

Definition 2.2. [3] Let X be a Banach space and $\mathcal{B}_X$ the family of bounded subset of X. A map

$$\mu : \mathcal{B}_X \to [0, \infty)$$

is called measure of noncompactness (MNC) defined on X if it satisfies the following

1. $\mu(A) = 0 \Leftrightarrow A$ is a precompact set.
2. $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$.
3. $\mu(A) = \mu\left(\overline{A}\right)$, $\forall A \in \mathcal{B}_X$.
4. $\mu(ConvA) = \mu(A)$.
5. $\mu(\lambda A + (1-\lambda) B) \leqslant \lambda\mu(A) + (1-\lambda)\mu(B)$, for $\lambda \in [0,1]$.
6. Let (A_n) be a sequence of closed sets from $\mathcal{B}_X$ such that $A_{n+1} \subseteq A_n$, $(n \geqslant 1)$ and $\lim\limits_{n\to\infty} \mu(A_n) = 0$. Then the intersection set $A_\infty = \bigcap\limits_{n=1}^{\infty} A_n$ is nonempty and A_∞ is precompact.

Let μ_0 be the sequential MNC generated by μ, that is, for any bounded subset $A \subset X$, then

$$\mu_0(A) = \sup\left\{\mu(x_n)_{n=1}^{\infty} \text{ is a sequence in } A\right\}.$$

The relation between μ_0 and μ is given by the following inequalities

$$\mu_0(A) \leqslant \mu(A) \leqslant 2\mu_0(A).$$

However, if X is a separable space then $\mu_0(A) = \mu(A)$.

Lemma 2.3 ([10])**.**

1. *Let $A \subseteq C(H;X)$ is bounded, then $\mu(A(t)) \leqslant \mu(A)$ for all $t \in H$, where $A(t) = \{y(t), y \in A\} \subset X$. Furthermore if A is equicontinuous on H, then $\mu(A(t))$ is continuous on H and $\mu(A) = \sup\{\mu(A(t)), t \in H\}$.*
2. *If $A \subset C(E;X)$ is bounded and equicontinuous, then*

$$\mu\left(\int_0^t A(s)\,ds\right) \leqslant \int_0^t \mu(A(s))\,ds,$$

for all $t \in E$, where $\int_0^t A(s)\,ds = \left\{\int_0^t x(s)\,ds : x \in A\right\}$.

Lemma 2.4 ([15])**.** *Let X be a Banach space and F a Caratheodory multivalued mapping. Let $\Phi : L^1(E;X) \to C(E;X)$ be linear continuous mapping. Then,*

$$\begin{aligned} \Phi \circ S_F : C(E;X) &\to \mathcal{P}_{cl;c}(C(E;X)) \\ u &\to (\Phi \circ S_F)\,u := \Phi(S_F(u)), \end{aligned}$$

is a closed graph operator in $C(E;X) \times C(E;X)$.

Definition 2.5 ([2])**.** A multivalued map $F : X \to \mathcal{P}(X)$ is called k-set contraction multivalued mapping if there exists a constant k, $0 \leqslant k < 1$ such that

$$\mu(FA) \leqslant k\mu(A) \text{ for any bounded } A \subset X.$$

If $\mu(FA) < k\mu(A)$, then F is a condensing multivalued mapping.

Theorem 2.6. *Let A be a closed convex and bounded subset of a Banach space X and let $F : A \to \mathcal{P}_{cl,cv}(A)$ be an upper semi-continuous and condensing multivalued mapping. Then, F has a fixed point point.*

The following results is due to Dhage [6].

Theorem 2.7. *Let A be a closed convex and bounded subset of a Banach space X and let $F : A \to \mathcal{P}_{cl,cv}(A)$ be an upper semi-continuous multivalued mapping such that*

$$\mu(FW) \leqslant \varphi(\mu(W)) \text{ for any bounded } W \subset A,$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function that satisfies $\varphi(t) < t$.
Then, F has a fixed point point and the set of fixed points is compact .

Lemma 2.8. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing and upper semi-continuous function. Then,*

$$\lim_{n\to\infty} \psi^n(t) = 0 \text{ for each } t > 0 \Leftrightarrow \psi(t) < t \text{ for any } t > 0.$$

In what follows, we confine ourselves only to the fixed point theory related to upper semicontinuous multi-valued mappings in Banach spaces. The first fixed point theorem in this direction is due to Kakutani–Fan [8] which is as follows.

Theorem 2.9. *Let A be a nonempty compact convex subset of a Hausdorff locally convex topological vector space E, and let $F : A \to \mathcal{P}_{cl,cv}(A)$ be an upper semicontinuous map. Then, F has a fixed point.*

3. Fixed point theorems for multivalued Meir–Keeler set contraction mappings

Definition 3.1. A Meir–Keeler condensing multivalued mapping if for each $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\varepsilon \leqslant \mu(A) < \varepsilon + \delta \Rightarrow \mu(FA) < \varepsilon.$$

Remark 3.2. The condensing multivalued mappings of Meir–Keeler type are more general than condensing mappings. Indeed, let F be a condensing mapping, that is,

$$\mu(FA) \leqslant k\mu(A) \text{ for any bounded } A \subset X.$$

Suppose for $\delta = \left(\frac{1}{k} - 1\right)\varepsilon$ that we have $\varepsilon \leqslant \mu(A) < \varepsilon + \delta$, then

$$\mu(FA) \leqslant k\mu(A) < \varepsilon + k\left(\frac{1}{k} - 1\right)\varepsilon = \varepsilon.$$

Thus, F is a Meir Keeler condensing multivalued mapping.

Theorem 3.3. *Let X be a Banach space and A be a nonempty closed, bounded and convex subset of X. Let $F : A \to \mathcal{P}_{cl,cv}(A)$ be multivalued upper semicontinuous mapping such that for any bounded $W \subset A$, we have*

$$\varepsilon \leqslant \mu(W) < \varepsilon + \delta \Rightarrow \mu(FW) < \varepsilon.$$

Then, F has at least one fixed point in A.

Proof. Obviously, if we have

$$\varepsilon \leqslant \mu(W) < \varepsilon + \delta \Rightarrow \mu(FW) < \varepsilon,$$

then

$$\mu(FA) < \mu(A).$$

Thus by Theorem 2.6, F has at least one fixed point.

Corollary 3.4. *Let X be a Banach space and $F : X \to \mathcal{P}(X)$ be multivalued mapping with convex values, closed graph and bounded range such that, for any bounded $A \subset X$, we have*

$$\mu(FA) \leqslant k\mu(A), \text{ for } 0 \leqslant k < 1.$$

Then, F has at least one fixed point in A.

4. Fixed point theorems for multivalued set contraction mappings of Caristi type

Theorem 4.1. *Let X be a Banach space and A be a nonempty closed, bounded and convex subset of X. Let $F : A \to \mathcal{P}_{cl,cv}(A)$ be multivalued upper semicontinuous mapping such that for any bounded $W \subset A$, we have*

$$\psi(\mu(FW)) \leqslant \psi(\mu(W)) - \varphi(\mu(W)), \tag{4.1}$$

where μ is an arbitrary measure of noncompactness and $\psi, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions such that φ is lower semi-continuous and ψ is continuous on $\mathbb{R}_+$. Moreover, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Then, F has at least one fixed point in A.

Proof. Define the sequence $W_0 = W$ and $W_{n+1} = \overline{co}(FW_n)$, clearly $(W_n)_{n\in\mathbb{N}}$ is a nonempty closed, bounded, convex sequence and

$$W_0 \subset W_1 \subset \cdots \subset W_n.$$

Since the sequence $(\mu(W_n))_{n\in\mathbb{N}}$ is decreasing and bounded below (since $\mu(W_n) > 0$, $\forall n \in \mathbb{N}$), then $(\mu(W_n))_{n\in\mathbb{N}}$ is a convergent sequence. Put $\lim_{n\to\infty} \mu(W_n) = l$.

In further, using properties of the measure of noncompactness we have,

$$\mu(W_{n+1}) = \mu(\overline{co}(FW_n)) = \mu(FW_n).$$

Then, in view of condition (4.1) we have

$$\begin{aligned} \psi(\mu(W_{n+1})) &= \psi(\mu(FW_n)) \\ &\leqslant \psi(\mu(W_n)) - \varphi(\mu(W_n)). \end{aligned}$$

By taking the limit sup we get

$$\lim_{n\to\infty} \sup \psi(\mu(W_{n+1})) \leqslant \lim_{n\to\infty} \sup \psi(\mu(W_n)) - \lim_{n\to\infty} \inf \varphi(\mu(W_n)).$$

Since ψ is continuous and φ is lower semi-continuous, we get

$$\psi(l) \leqslant \psi(l) - \varphi(l).$$

Fellows that $\varphi(l)$ must be null, which means that $l = 0$. Thus

$$0 = \lim_{n\to\infty} \sup \mu(W_n) = \lim_{n\to\infty} \inf \mu(W_n) = \lim_{n\to\infty} \mu(W_n).$$

Hence, using property 6. of measure of noncompactness we get $W_\infty = \bigcap_n W_n$ is compact. Then, F has at least one fixed point. □

5. Existence of fixed points for multivalued power set contraction mappings

Theorem 5.1. *Let A be a nonempty closed, bounded and convex subset of a Banach space X and $N : A \to \mathcal{P}_{cl,cv}(A)$ be a k-set contraction mapping on A. Then, N^n (for an integer $n > 0$) is a k^n-set contraction on A.*

Proof. Let A be a nonempty closed, bounded and convex subset of X, then for any bounded bounded $W \subset A$,

$$\begin{aligned} \mu(N^n W) &= \mu\left(N\left(N^{n-1}W\right)\right) \\ &\leqslant k\mu\left(N^{n-1}W\right) \\ &\leqslant k^2\mu\left(N^{n-2}W\right) \\ &\vdots \\ &\leqslant k^n\mu(W). \end{aligned}$$

Since $0 \leqslant k < 1$, hence $0 \leqslant k^n < 1$ and so N^n is also a $k-$set contraction mapping. □

Remark 5.2. The inverse is not true that is if N^n is a k-set contraction mapping then N could be not a k-set contraction mapping.

Theorem 5.3. *Let A be a nonempty closed, bounded and convex subspace of a Banach space X and $N : A \to \mathcal{P}_{cl,cv}(A)$ be an upper semi-continuous multivalued mapping such that for any $n \geqslant 1$ we have $N^n(conv(W)) \subseteq conv(N^n W)$ and*

$$\mu(N^n W) \leqslant k_n \mu(W), \text{ for any bounded } W \subset A. \tag{5.1}$$

where $k_n \to 0$, $n \to +\infty$. Then, there exists at least one x such that $x \in Nx$.

Proof. Let the iteration $W_0 = W$ and $W_n = \overline{conv}(NA_{n-1})$. Obviously (A_n) is a sequence of nonempty closed, bounded and convex subsets of A.

It is clear that $(A_n)_n$ is decreasing.

Then, by using the properties of the measure of noncompactness, we get

$$\begin{aligned} \mu(W_n) &= \mu(conv(NW_{n-1})) \\ &\leqslant \mu(conv(NW_{n-1})) = \mu(NW_{n-1}) \\ &\leqslant \mu(N(conv(NW_{n-2}))) \\ &\leqslant \mu(N(conv(NW_{n-2}))) \\ &\leqslant \mu\left(N^2(W_{n-2})\right) \end{aligned}$$

Repeating this process many times we get

$$\mu\left(W_n\right) \leqslant \mu\left(N^n\left(W_0\right)\right).$$

Using Inequality 5.1. we get $\mu\left(W_n\right) \leqslant \mu\left(N^n\left(W_0\right)\right) \leqslant k_n \mu\left(W_0\right).$

By taking the limit, we get $\lim_{n\to\infty} \mu\left(W_n\right) = 0$, which implies that W_∞ is compact. Hence N has at least one fixed point in $W_\infty \subset A$. □

6. Application to Evolution differential inclusions with nonlocal condition

The multi-valued fixed point theorems of this paper can have some nice applications to differential and integral inclusions as an example we choose to provide an application for Theorem 3.3. One can notice that other applications can be given by changing the contractive condition which the mappings is supposed to satisfy.

Let following evolution differential inclusions with nonlocal conditions

$$y'(t) \in A(t)y(t) + F(t, y\left(t\right)), \quad t \in J := [0, +\infty) \tag{6.1}$$

$$y(0) = \varphi(y), \tag{6.2}$$

where F is an upper Caratheodory multimap, $\varphi : C\left(J, X\right) \to X$ is a given X-valued function. $\{A\left(t\right) : t \in J\}$ is a family of linear closed unbounded operators on X with domain $D(A(t))$ independent of t that generate an evolution system of operators $\{U\left(t, s\right) : t, s \in \Delta\}$ with $\Delta = \{(t, s) \in J \times J : 0 \leqslant s \leqslant t < \infty\}$.

The main work for this section is to study the existence of mild solutions for this non-local inclusion.

Before we start studying this problem we recall some concepts and results that will be needed through the section.

Define the set

$$S_F\left(y\right) = \left\{f \in L^1\left(J, X\right) : f\left(t\right) \in F\left(t, y\left(t\right)\right)\right\}.$$

Definition 6.1. A mapping $F : J \times C(J, X) \longrightarrow \mathcal{P}_{cp,cv}(X)$ is said to be an upper Carathéodory multivalued map if it satisfies,

- (i) $x \mapsto F(t, x)$ is upper semi-continuous (with respect to the metric H_d) for almost all $t \in J$.
- (ii) $t \mapsto F(t, x)$ is measurable for each $x \in C(J, X)$.

Definition 6.2. A family $\{U(t, s)\}_{(t,s)\in\Delta}$ of bounded linear operators $U(t, s) : X \to X$ where $(t, s) \in \Delta := \{(t, s) \in J \times J : 0 \leqslant s \leqslant t < +\infty\}$ is called en evolution system if the following properties are satisfied,

1. $U(t, t) = I$ where I is the identity operator in X and $U(t, s)\ U(s, \tau) = U(t, \tau)$ for $0 \leqslant \tau \leqslant s \leqslant t < +\infty$,
2. The mapping $(t, s) \to U(t, s)\ y$ is strongly continuous, that is, there exists a constant $M > 0$ such that

$$\|U(t, s)\| \leqslant M \quad \text{for any } (t, s) \in \Delta.$$

An evolution system $U(t,s)$ is said to be compact if $U(t,s)$ is compact for any $t-s>0$. $U(t,s)$ is said to be equicontinuous if $\{U(t,s)x : x\in M\}$ is equicontinuous at $0\leqslant s<t\leqslant b$ for any bounded subset $M\subset X$. Clearly, if $U(t,s)$ is a compact evolution system, it must be equicontinuous. The inverse is not necessarily true.

More details on evolution systems and their properties could be found on the books of Ahmed [1], Engel and Nagel [7] and Pazy [13].

Definition 6.3. We say that the function $y(t)\in C(J,X)$ is a mild solution of the evolution system (6.1) − (6.2) if it satisfies the following integral equation

$$y(t)=U(t,0)\ \varphi(y)+\int_0^t U(t,s)\ f(s)\ ds, \tag{6.3}$$

for all $t\in\mathbb{R}_+$ and $f\in S_F(y)$.

Assume the following hypothesis which are needed thereafter :

- $(H1)$ $\{A(t) : t\in J\}$ is a family of linear operators. $A(t) : D(A)\subset X\to X$ generates an equicontinuous evolution system $\{U(t,s) : (t,s)\in\Delta\}$ and
$$|U(t,s)|\leqslant M.$$
- $(H2)$ The multifunction $F : J\times C(J;X)\longrightarrow\mathcal{P}_{cl,cv}(X)$ is an upper Carathéodory and $\varphi : C(J;X)\to X$ is continuous, if we have for any $\varepsilon>0$ there exists $\delta>0$ such that
$$\varepsilon\leqslant\mu(W)<\varepsilon+\delta \text{ for any bounded } W\subset A$$
implies
$$\mu(\varphi(W))<\frac{\varepsilon}{2M} \text{ and } \mu(F(t,W))<\frac{\varepsilon}{2Mt} \text{ for any } t\in J.$$
- $(H3)$ There exists a constant $r>0$ such that
$$M\left[\|\varphi(y)\|+\{\|f(t)\|_1 : f\in S_F(y),y\in A_0\}\right]\leqslant r$$
where, $A_0=\{y\in C(J;X) : \|y(t)\|\leqslant r \text{ for all } t\in J\}$.

Theorem 6.4. *Under the assumptions* $(H1)-(H3)$ *the non local problem* (*6.3*)−(*6.2*) *has at least one mild solution in the space* $C(J,X)$.

Proof. To solve problem (6.3) − (6.2) we transform it to the following fixed-point problem.

Consider the multivalued operator $N : C(J;X)\to\mathcal{P}(C(J;X))$ defined by

$$N(y)=\left\{h\in C(J;X) : h(t)=U(t,0)\varphi(y)+\int_0^t U(t,s)\ f(s)\ ds, \text{ with } f\in S_F(y)\right\}.$$

We can notice that fixed points of the operator N are mild solutions of problem (6.3) − (6.2).

Clearly for each $y\in C([-r,+\infty);X)$, the set $S_F(y)$ is nonempty since, by $(H2)$, F has a measurable selection (see [4]).

To prove that N has a fixed point , we need to satisfy all the conditions of one of above theorems, for example let choose Theorem 5.3.

Let $A_0 = \{y \in C(J;X) : \|y(t)\| \leqslant r \text{ for all } t \in J\}$. Obviously, A_0 is closed, bounded and convex.

To show that $NA_0 \subseteq A_0$, we need first to prove that the family

$$\left\{ \int_0^t U(t,s)\ f(s)\ ds\ : f \in S_F(y) \text{ and } y \in A_0 \right\}$$

is equicontinuous for $t \in J$ that is all the functions are continuous and they have equal variation over a given neighborhood.

In view of $(H1)$ we have that functions in the set $\{U(t,s) : (t,s) \in \Delta\}$ are equicontinuous, (i,e) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-\tau| < \delta$ implies $\|U(t,s) - U(\tau,s)\| < \varepsilon$ for all $U(t,s) \in \{U(t,s) : (t,s) \in \Delta\}$

Then, given some $\varepsilon > 0$ let $\delta = \frac{\varepsilon'}{\varepsilon \|f\|_\infty}$ such that $|t-\tau| < \delta$, we have

$$\left| \int_0^t U(t,s)\ f(s)\ ds - \int_0^\tau U(\tau,s)\ f(s)\ ds \right| \leqslant \int_\tau^t |U(t,s) - U(\tau,s)|\, |f(s)|\, ds.$$

Regarding the fact that $\{U(t,s) : (t,s) \in \Delta\}$ is equicontinuous then

$$\begin{aligned} \left| \int_0^t U(t,s)\ f(s)\ ds - \int_0^\tau U(\tau,s)\ f(s)\ ds \right| &\leqslant \varepsilon \|f\|_\infty |t-\tau| \\ &< \varepsilon \|f\|_\infty \frac{\varepsilon'}{\varepsilon \|f\|_\infty} = \varepsilon'. \end{aligned}$$

Hence, we conclude that $\left\{ \int_0^t U(t,s)\ f(s)\ ds\ : f \in S_F(y) \text{ and } y \in A_0 \right\}$ is equicontinuous for $t \in J$.

Now, let show that $NA_0 \subseteq A_0$. Let for $t \in J$,

$$\begin{aligned} |h(t)| &= \left| U(t,0)\,\varphi(y) + \int_0^t U(t,s)\ f(s)\ ds \right| \\ &\leqslant |U(t,0)\,\varphi(y)| + \int_0^t |U(t,s)\ f(s)|\, ds \\ &\leqslant M\,\|\varphi(y)\| + M\,\|f\|_1 \\ &= M\,[\|\varphi(y)\| + \|f\|_1] \leqslant r, \end{aligned}$$

thus $NA_0 \subseteq A_0$.

In further it is easy to see that N has convex valued.

Now let show that N has a closed graph, let $y_n \to y$ and $h_n \to h$ such that $h_n(t) \in N(y_n)$ and let show that $h(t) \in N(y)$.

Then, there exists a sequence $f_n \in S_F(y_n)$ such that

$$h_n(t) = U(t,0)\,\varphi(y_n) + \int_0^t U(t,s) f_n(s) ds.$$

Consider the linear operator $\Phi : L^1(J;X) \to C(J;X)$ defined by

$$\Phi f(t) = \int_0^t U(t,s) f_n(s) ds.$$

Clearly, Φ is linear and continuous. Then from Lemma 2.4. we get that $\Phi \circ S_F(y)$ is a closed graph operator. In further, we have

$$h_n(\cdot) - U(t,0)\varphi(y_n) \in \Phi \circ S_{F,y}.$$

Since $y_n \to y$ and $h_n \to h$, then

$$h(\cdot) - U(t,0)\varphi(y_n) \in \Phi \circ S_{F,y}.$$

That is, there exists a function $f \in S_F(y)$ such that

$$h(t) = U(t,0)\varphi(y) + \int_0^t U(t,s)f(s)ds.$$

Therefore N has a closed graph, hence N has closed values on $C(J;X)$.

Let W be a bounded subset of A such that

$$\varepsilon \leqslant \mu(W) < \varepsilon + \delta.$$

We know that the family $\left\{\int_0^t U(t,s)f(s)ds, f \in S_F(W(t))\right\}$ is equicontinuous, hence by Lemma 2.3, we have

$$\begin{aligned}
\mu\left(\int_0^t U(t,s)f(s)ds,\ f \in S_F(W(t))\right) &\leqslant \int_0^t \mu(U(t,s)f(s),\ f \in S_F(W(t)))\,ds \\
&\leqslant M\int_0^t \mu(f(s),\ f \in S_F(W(t)))\,ds \\
&\leqslant Mt\mu(F(t,W(t))).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mu(NW) &= \mu N\left(U(t,0)\varphi(W(t)) + \int_0^t U(t,s)f(s)ds,\ f \in S_F(W(t))\right) \\
&\leqslant \mu(U(t,0)\varphi(W(t))) + \mu\left(\int_0^t U(t,s)f(s)ds,\ f \in S_F(W(t))\right) \\
&\leqslant M\mu(\varphi(W(t))) + Mt\mu(F(t,W(t))).
\end{aligned}$$

In view of $(H2)$, we get

$$\mu(NW(t)) \leqslant M\frac{\varepsilon}{2M} + Mt\frac{\varepsilon}{2Mt} = \varepsilon.$$

Therefore, for $\varepsilon \leqslant \mu < \varepsilon+\delta$ we obtained $\mu(NW(t)) \leqslant \varepsilon$. Thus regarding Theorem 3.3, N has at least one fixed point, hence the problem (6.1) – (6.2) has at least one mild solution in the space $C(J,X)$.

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