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ON SKEW $[m, C]$-SYMMETRIC OPERATORS

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Dedicated to the memory of Professor Takayuki Furuta with deep sorrow

Communicated by D. S. Djordjević

Abstract. In this paper, first we characterize the spectra of skew $[m, C]$-symmetric operators and we also prove that if operators T and S are C-doubly commuting operators, T is a skew $[m, C]$-symmetric operator and Q is an n-nilpotent operator, then $T + Q$ is a skew $[m + 2n - 2, C]$-symmetric operator. Finally, we show that if T is skew $[m, C]$-symmetric and S is $[n, D]$-symmetric, then $T \otimes S$ is skew $[m + n - 1, C \otimes D]$-symmetric.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\, , \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. J. Agler and M. Stankus studied m-isometric operators ([1]). L.W. Helton introduced m-symmetric operators for the study of Jordan operators ([6]). For an operator $T \in B(\mathcal{H})$, the operator $\alpha_m(T)$ is defined by

$$\alpha_m(T) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^j \quad (m \in \mathbb{N}),$$

where $\mathbb{N}$ is the set of all natural numbers. In particular, if T is normal, then $\alpha_m(T) = (T^* - T)^m$. An operator $T \in B(\mathcal{H})$ is said to be *m-symmetric* if $\alpha_m(T) = 0$. Hence it is clear that if T is normal and m-symmetric, then T

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is Hermitian. Since, $\alpha_{m+1}(T) = T^* \cdot \alpha_m(T) - \alpha_m(T) \cdot T$, it holds that if T is m-symmetric, then T is n-symmetric for all $n \geq m$. S. A. McCullough and L. Rodman proved that if T is m-symmetric and m is even, then T is always $(m-1)$-symmetric (Theorem 3.4 of [9]). For an operator $T \in B(\mathcal{H})$, the spectrum, the point spectrum, the approximate point spectrum and the surjective spectrum of T are denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$ and $\sigma_s(T)$, respectively. It's well known that $\sigma(T) = \sigma_a(T) \bigcup \sigma_s(T)$ and $\sigma_a(T)^* = \sigma_s(T^*)$, where $A^* = \{\overline{a} : a \in A \subset \mathbb{C}\}$.

Recently, C. Gu and M. Stankus ([5]) showed interesting properties of m-symmetric operators. An antilinear operator C on $\mathcal{H}$ is said to be a *conjugation* if C satisfies $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$, where I is the identity operator on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be a *complex symmetric* operator if $CTC = T^*$ for some conjugation C. An operator $T \in B(\mathcal{H})$ is said to be a *skew symmetric operator* if $CTC = -T^*$ for some conjugation C. For an operator $T \in B(\mathcal{H})$ and a conjugation C, let $A = \frac{1}{2}(T + CT^*C)$ and $B = \frac{1}{2}(T - CT^*C)$. Then it is easy to see that A is complex symmetric, B is skew symmetric and $T = A + B$. In [8], C. G. Li and S. Zhu showed Structure Theorem for skew symmetric normal operators as follows:

Theorem 1.1. (*Theorem 1.10,* [8]) *Let $T \in B(\mathcal{H})$ be normal. Then the following are equivalent:*

(1) *T is skew symmetric;*

(2) *$T_{|\ker(T)^\perp} \simeq N \oplus (-N)$, where N is a normal operator on some Hilbert space $\mathcal{K}$.*

See [2], [4], [7] and [8] for examples and details of conjugations, complex symmetric operators and skew symmetric operators. In [7], S. Jung, E. Ko, M. Lee, and J. E. Lee studied spectral properties of complex symmetric operators and they proved the following.

Proposition 1.2. (Lemma 3.21, [7]). *For $T \in B(\mathcal{H})$ and a conjugation C it holds*

$\sigma(CTC) = \sigma(T)^*, \sigma_p(CTC) = \sigma_p(T)^*, \sigma_a(CTC) = \sigma_a(T)^*$ *and* $\sigma_s(CTC) = \sigma_s(T)^*$.

Remark 1.3. In the above proposition, there is no relation between T and CTC.

Definition 1.4. For $T \in B(\mathcal{H})$ and a conjugation C, set

$$\zeta_m(T; C) := \sum_{j=0}^{m} \binom{m}{j} CT^{m-j}C \cdot T^j.$$

An operator T is said to be *skew* $[m, C]$*-symmetric* if $\zeta_m(T; C) = 0$.

It holds that $CTC \cdot \zeta_m(T; C) + \zeta_m(T; C) \cdot T = \zeta_{m+1}(T; C)$. Hence, if T is skew $[m, C]$-symmetric, then T is skew $[n, C]$-symmetric for all $n \geq m$. In [2], M. Chō, Dragan S. Djordjevic, Ji Eun Lee and B. Načevska Nastovska have been studied properties of the approximate point spectra of skew $[m, C]$-symmetric operators and others.

If T is skew $[1, C]$-symmetric, then it holds $CTC = -T$. For $A \subset \mathbb{C}$, let $-A = \{-a : a \in A\}$. By Proposition 1.2, if T is skew $[1, C]$-symmetric, then it clearly holds

$$\sigma(T)^* = -\sigma(T), \quad \sigma_p(T)^* = -\sigma_p(T), \quad \sigma_a(T)^* = -\sigma_a(T) \text{ and } \sigma_s(T)^* = -\sigma_s(T).$$

Throughout this paper, let C be a conjugation on $\mathcal{H}$ and m, n be natural numbers. An operator $Q \in B(\mathcal{H})$ is said to be an n-nilpotent operator if $Q^n = 0$.

2. Main results

First we show the following result for skew $[m, C]$-symmetric operators.

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be skew $[m, C]$-symmetric. Then the following statements hold:*

$$\sigma(T)^* = -\sigma(T), \quad \sigma_p(T)^* = -\sigma_p(T), \quad \sigma_a(T)^* = -\sigma_a(T) \text{ and } \sigma_s(T)^* = -\sigma_s(T).$$

Proof. Proof of $\sigma_a(T)^* = -\sigma_a(T)$. Let $a \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - a)x_n \to 0$ as $n \to \infty$. Since

$$0 = \sum_{j=0}^{m} \binom{m}{j} CT^{m-j}C \cdot T^j x_n = (CTC + a)^m x_n + \sum_{j=1}^{m} \binom{m}{j} CT^{m-j}C \cdot (T^j - a^j)x_n,$$

it holds that $\lim_{n\to\infty} (CTC + a)^m x_n = 0$. So, since $-a \in \sigma_a(CTC) = \sigma_a(T)^*$, we get $-\sigma_a(T) \subset \sigma_a(T)^*$, and also $-\sigma_a(T)^* \subset \sigma_a(T)$, which proves $\sigma_a(T)^* = -\sigma_a(T)$. Furthermore, it is clear that $\sigma_p(T)^* = -\sigma_p(T)$.

Proof of $\sigma_s(T)^* = -\sigma_s(T)$. Having in mind that $\sigma_s(T)^* = \sigma_a(T^*)$ and for $a \in \sigma_a(T^*)$, there exists a sequence $\{x_n\}$ of unit vectors such that $(T^* - a)x_n \to 0$ as $n \to \infty$.

Since, $0 = \sum_{j=0}^{m} \binom{m}{j} CT^{m-j}C \cdot T^j$, it holds that $0 = \sum_{j=0}^{m} \binom{m}{j} T^{*j} \cdot CT^{*m-j}C$. Then multiplying it by C from both sides, we have

$$0 = \sum_{j=0}^{m} \binom{m}{j} CT^{*j}C \cdot T^{*m-j}.$$

Hence,

$$\begin{aligned} 0 &= \sum_{j=0}^{m} \binom{m}{j} CT^{*j}C \cdot T^{*m-j} x_n \\ &= (CT^*C + a)^m x_n + \sum_{j=0}^{m} \binom{m}{j} CT^{*j}C \cdot (T^{*m-j} - a^{m-j}) x_n. \end{aligned}$$

Therefore, since $\lim_{n\to\infty} (CT^*C + a)^m x_n = 0$, we have $-a \in \sigma_a(CT^*C) = \sigma_a(T^*)^* = \sigma_s(T)$ and $-\sigma_s(T)^* \subset \sigma_s(T)$. So, we have $\sigma_s(T)^* \subset -\sigma_s(T)$ and also it holds that $\sigma_s(T) \subset -\sigma_s(T)^*$. Therefore, $\sigma_s(T)^* = -\sigma_s(T)$ holds. This implies $\sigma(T)^* = -\sigma(T)$. □

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be skew $[m, C]$-symmetric.*

(1) *Then T^* is skew $[m, C]$-symmetric.*
(2) *If there exists T^{-1}, then T^{-1} is also skew $[m, C]$-symmetric.*
(3) *If T_n are skew $[m, C]$-symmetric and $\lim_{n\to\infty} T_n = T$, then T is skew $[m, C]$-symmetric.*

Proof. Proof of (1). Since

$$0 = \Big(\sum_{j=0}^{m} \binom{m}{j} CT^{m-j}C \cdot T^j\Big)^* = \sum_{j=0}^{m} \binom{m}{j} T^{*j} \cdot CT^{*m-j}C,$$

$$0 = C\Big(\sum_{j=0}^{m} \binom{m}{j} T^{*j} \cdot CT^{*m-j}C\Big)C = \sum_{j=0}^{m} \binom{m}{j} CT^{*j}C \cdot T^{*m-j} = \zeta_m(T^*, C).$$

It completes (1).
Proof of (2). Multiplying by C from the left side in the equation $\zeta_m(T; C) = 0$, i.e., $0 = \sum_{j=0}^{m} \binom{m}{j} CT^{m-j}C \cdot T^j$, we have

$$0 = \sum_{j=0}^{m} \binom{m}{j} T^{m-j}C \cdot T^j.$$

Then again, multiplying by T^{-m} from both sides in the last equation, it follows that $0 = \sum_{j=0}^{m} \binom{m}{j} T^{-j}C \cdot T^{-m+j}$. Now, multiplying by C from the left side of this equation we get

$$0 = \sum_{j=0}^{m} \binom{m}{j} CT^{-j}C \cdot T^{-m+j} = \sum_{j=0}^{m} \binom{m}{j} C(T^{-1})^jC \cdot (T^{-1})^{m-j}.$$

Hence (2) has been proved.
Proof of (3). Since, $\lim_{n\to\infty} T_n^j = T^j$ and $\lim_{n\to\infty} CT_n^jC = CT^jC$ for any $j \in \mathbb{N}$, we have $0 = \zeta_m(T_n; C) \longrightarrow \zeta_m(T; C)$, as $n \to \infty$. Therefore, we have $\zeta_m(T; C) = 0$. □

Theorem 2.3. *If Q is m-nilpotent, then Q is skew $[2m-1, C]$-symmetric for any conjugation C.*

Proof. It holds

$$\zeta_{2m-1}(Q; C) = \sum_{j=0}^{2m-1} \binom{2m-1}{j} CQ^{2m-1-j}C \cdot Q^j.$$

(1) If $j \geq m$, then $Q^j = 0$. (2) If $j \leq m-1$, then since $2m-1-j \geq 2m-1-(m-1) = m$, $CQ^{2m-1-j}C = 0$. Hence it completes the proof. □

For the study of the sum $T+S$, we need the following property.

Definition 2.4. Operators T and S are said to be C-*doubly commuting* if $TS = ST$ and $CSC \cdot T = T \cdot CSC$.

From the equation

$$(a+x+b+y)^m = ((a+b)+(x+y))^m = \sum_{j=0}^{m} \binom{m}{j} (a+b)^{m-j} \cdot (x+y)^j,$$

if T and S are C-doubly commuting, then the following equation holds

$$\zeta_m(T+S;C) = \sum_{j=0}^{m} \binom{m}{j} \zeta_{m-j}(T;C) \cdot \zeta_j(S;C). \tag{2.1}$$

Using the equation (2.1), the next Theorem is proved.

Theorem 2.5. *Let T be skew $[m,C]$-symmetric and S be skew $[n,C]$-symmetric. If T and S are C-doubly commuting, then $T+S$ is skew $[m+n-1,C]$-symmetric.*

Proof. By (2.1) and similar proof as of Theorem 2.3, the result follows. ☐

So we have the following corollary. Since the proof is easy, it's omitted.

Corollary 2.6. *Let T be skew $[m,C]$-symmetric and Q be n-nilpotent. If T and Q are C-doubly commuting, then $T+Q$ is skew $[m+2n-2,C]$-symmetric.*

Remark 2.7. Let $\mathcal{H} = \mathbb{C}^2$, $C\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \overline{y} \\ \overline{x} \end{pmatrix}$ and, for a non-zero real number a, let $R = \begin{pmatrix} i & a \\ 0 & i \end{pmatrix}$. Then, it is easy to see that R is skew $[3,C]$-symmetric. Now, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$. Then T and S are skew $[3,C]$-symmetric. And we have $TS = ST, CSC \cdot T \neq T \cdot CSC$ and $T+S = \begin{pmatrix} i & 2 \\ 0 & i \end{pmatrix}$. Hence $T+S$ is skew $[3,C]$-symmetric and also skew $[3+2\cdot 3-2,C]$-symmetric, because $7>3$. Unfortunately, in this moment, we do not have a nice counterexample for the necessity of C-doubly commutingness.

For the study of properties of the product TS of operators T and S, we need the following class of operators.

Definition 2.8. For an operator T and a conjugation C, set

$$\alpha_m(T;C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^j.$$

T is said to be $[m,C]$-*symmetric* if $\alpha_m(T;C) = 0$.

Having in mind that

$$(ax+by)^m = ((a+b)x - b(x-y))^m = \sum_{j=0}^{m} (-1)^j \, (a+b)^{m-j} \cdot b^j \cdot x^{m-j} \cdot (x-y)^j,$$

if T and S are C-doubly commuting, the following holds

$$\zeta_m(TS;C) = \sum_{j=0}^{m} (-1)^j \zeta_{m-j}(T;C) \cdot T^j \cdot CS^{m-j}C \cdot \alpha_j(S;C). \tag{2.2}$$

So the next Theorem holds.

Theorem 2.9. *Let T be skew $[m,C]$-symmetric and S be $[n,C]$-symmetric. If T and S are C-doubly commuting, then TS is skew $[m+n-1,C]$-symmetric.*

Proof. Using (2.2), it holds that

$$\zeta_{m+n-1}(TS;C) = \sum_{j=0}^{m+n-1} (-1)^j \zeta_{m+n-1-j}(T;C) \cdot T^j \cdot CS^{m+n-1-j}C \cdot \alpha_j(S;C).$$

(1) If $j \geq n$, then $\alpha_j(S;C) = 0$. (2) If $j \leq n-1$, then $\zeta_{m+n-1-j}(T;C) = 0$. Therefore the proof is completed. □

Remark 2.10. In general, it does not hold that if T is skew $[m,C]$-symmetric, then T^2 is skew $[n,C]$-symmetric for some n. For example, let $T = \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}$. Then for the conjugation C such that $C\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \overline{y} \\ \overline{x} \end{pmatrix}$, T is skew $[1,C]$-symmetric. But since $T^2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$, T^2 is symmetric, i.e., it is not skew symmetric.

Finally we study the tensor product $T \otimes S$ according to B. Duggal [3]. Let $\mathcal{H}\overline{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H}\otimes\mathcal{H}$ of $\mathcal{H}$ with $\mathcal{H}$. For $T, S \in \mathcal{B}(\mathcal{H})$, let $T \otimes S \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})$ denote the tensor product on the Hilbert space $\mathcal{H}\overline{\otimes}\mathcal{H}$, when $T \otimes S$ is defined as follows

$$\langle T \otimes S(\xi_1 \otimes \eta_1),\ (\xi_2 \otimes \eta_2)\rangle = \langle T\xi_1,\ \xi_2\rangle\langle S\eta_1,\ \eta_2\rangle.$$

See the details by S. R. Garcia and M. Putinar p.1312 in [4].

We also have the following result.

Theorem 2.11. *Let T be skew $[m,C]$-symmetric and S be $[n,D]$-symmetric, then $T \otimes S$ is skew $[m+n-1, C \otimes D]$-symmetric.*

Proof. Let C and D be conjugations, then it is easy to see that $C \otimes D$ is a conjugation. Also, it is obvious that, if T is skew $[m,C]$-symmetric and S is skew $[n,D]$-symmetric, then $T \otimes I$ is skew $[m, C \otimes D]$-symmetric and $I \otimes S$ is $[n, C \otimes D]$-symmetric, too. Hence, $T \otimes I$ and $I \otimes S$ are $C \otimes D$-doubly commuting and since $(T \otimes I) \cdot (I \otimes S) = T \otimes S$, by Theorem 2.9 the result follows. □

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