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VARIANTS OF WEYL'S THEOREM FOR DIRECT SUMS OF CLOSED LINEAR OPERATORS

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ABSTRACT. If T is an operator with compact resolvent and S is any densely defined closed linear operator, then the orthogonal direct sum of T and S satisfies various Weyl type theorems if some necessary conditions are imposed on the operator S. It is shown that if S is isoloid and satisfies Weyl's theorem, then $T \oplus S$ satisfies Weyl's theorem. Analogous result is proved for a-Weyl's theorem. Further, it is shown that Browder's theorem is directly transmitted from S to $T \oplus S$. The converse of these results have also been studied.

1. INTRODUCTION

Let H and K be infinite dimensional separable complex Hilbert spaces and $C(H)$ be the set of all closed linear operators T from domain $\mathcal{D}(T) \subseteq H$ to H. By $\mathcal{N}(T)$ and $\mathcal{R}(T)$ we denote the null space and range of an operator T, respectively. We call an operator $T \in C(H)$ *upper semi-Fredholm* (respectively, *lower semi-Fredholm*) if $\mathcal{R}(T)$ is closed and nullity of T, $\alpha(T) = \dim \mathcal{N}(T) < \infty$ (respectively, defect of T, $\beta(T) = \operatorname{codim} \mathcal{R}(T) < \infty$). A *semi-Fredholm operator* is an upper or lower semi-Fredholm operator. If T is both upper and lower semi-Fredholm, that is, if both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm operator*. By $SF_+(H)$ (respectively, $SF_-(H)$) we denote the class of all upper (respectively, lower) semi-Fredholm operators. For $T \in SF_+(H) \cup SF_-(H)$, index of T is defined as $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. An operator $T \in C(H)$ is called

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Weyl if it is Fredholm of index 0 and the *Weyl spectrum* of T is defined as $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$. We shall use the following notations:

$$SF_+^-(H) = \{T \in C(H) \ : \ T \in SF_+(H) \text{ and } \operatorname{ind}(T) \leqslant 0\}$$

$$SF_-^+(H) = \{T \in C(H) \ : \ T \in SF_-(H) \text{ and } \operatorname{ind}(T) \geqslant 0\}$$

and these operators generate the following spectra

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_+^-(H)\}$$

$$\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_-^+(H)\}.$$

The *ascent* $p(T)$ and *descent* $q(T)$ of an operator $T \in C(H)$ are defined as follows:

$$p(T) = inf\{n : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$$

$$q(T) = inf\{n : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}.$$

An operator $T \in C(H)$ is *Browder* if T is a Fredholm operator with ascent $p(T)$ and descent $q(T)$ both finite. We denote by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ not Browder$\}$.

Clearly, $\sigma_W(\mathrm{T}) \subseteq \sigma_b(\mathrm{T})$.

Let $\sigma(T), \sigma_a(T)$ and $\rho(T)$ denote the spectrum, approximate spectrum and the resolvent set of T, respectively. By $\operatorname{iso}\sigma(T)$ and $\operatorname{iso}\sigma_a(T)$ we denote the isolated points of $\sigma(T)$ and $\sigma_a(T)$, respectively. It is well known that the resolvent operator $R_\lambda(T) = (T - \lambda I)^{-1}$ is an analytic operator-valued function for all $\lambda \in \rho(T)$ and the points of $\operatorname{iso}\sigma(T)$ are either poles or essential singularities of $R_\lambda(T)$. For $T \in C(H)$ with $\rho(T) \neq \emptyset$, $\lambda \in \operatorname{iso}\sigma(T)$ is said to be a *pole of order p* if p $= p(T - \lambda I) < \infty$ and $q(T - \lambda I) < \infty$ ([6]). Let $\pi_o(T)$ denote the set of all poles of finite multiplicity.

Let $E_o(T)$ and $E_o^a(T)$ denote the set of all eigenvalues of finite multiplicities in $\operatorname{iso}\sigma(T)$ and $\operatorname{iso}\sigma_a(T)$, respectively. If $T \in C(H)$, then T satisfies:

(i) Weyl's Theorem if $\sigma(T) \setminus \sigma_W(T) = E_o(T)$.
(ii) a-Weyl's Theorem if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_o^a(T)$.
(iii) Browder's Theorem if $\sigma(T) \setminus \sigma_W(T) = \pi_o(T)$.
(iv) a-Browder's Theorem if $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$.

An operator T is called *isoloid* if each $\lambda \in \operatorname{iso}\sigma(T)$ is an eigenvalue of T and *a-isoloid* if each $\lambda \in \operatorname{iso}\sigma_a(T)$ is an eigenvalue of T.

An important property of closed linear operators in Fredholm theory is the single valued extension property (SVEP). We mainly concern with the SVEP at a point, localized version of SVEP introduced by Finch [4], and relate it to the finiteness of the ascent of a closed linear operator. Let $T : \mathcal{D}(T) \subset H \to H$ be a closed linear mapping and let λ_o be a complex number. The operator T has the *single valued extension property* (SVEP) at λ_o if $f = 0$ is the only solution to $(T - \lambda I)f(\lambda) = 0$ that is analytic in every neighborhood of λ_o. Also, T has SVEP if it has this property at every point λ_o in the complex plane.

M. Berkani & H. Zariouh [2], B. P. Duggal & C. S. Kubrusly [3] and W. Y. Lee [7] have given sufficient conditions on the direct summands to ensure that Weyl-type theorems hold for the direct sum when both the operators are *bounded*. We aim to extend this study of direct sums of bounded linear operators to the classes of unbounded operators. A particularly simple yet important class of operators is the class of *unbounded operators with compact resolvent*. Important examples include the diagonal operator and most of the differential operators.

The following is the definition of unbounded operators with compact resolvent. Using the Spectral Mapping theorem for resolvents, several spectral properties for such class of operators follow analogously from the spectral properties of compact operators.

Definition 1.1. [5] Let $T \in C(H)$ be such that the resolvent operator $R_\lambda(T) = (T - \lambda I)^{-1}$ exists and is compact for some λ. Then the spectrum of T consists entirely of isolated eigenvalues of finite multiplicities, and $R_\lambda(T)$ is compact for every $\lambda \in \rho(T)$. Such an operator is called an *operator with compact resolvent*. [see Kato [5]]

In this paper, we take one of the summands T to be a closed linear operator with additional condition that it is an operator with compact resolvent and we shall give sufficient conditions on densely defined closed linear operator S to ensure that $T \oplus S$ satisfies various Weyl-type theorems. Throughout the article, we assume that $\rho(S) \neq \emptyset$. In the second section we are mainly concerned with the Weyl's and a-Weyl's Theorem and in the third section we deal with the Browder's Theorem for direct sums of unbounded operators. Example has been given to illustrate the results proved in this paper.

2. Weyl's and a-Weyl's Theorem for Direct sums of operators

We begin this section with the following theorem, which is instrumental in proving all the subsequent theorems.

Definition 2.1. [5, Ch IV, §1] Let $T \in C(H)$. Let A be an operator such that $\mathcal{D}(T) \subset \mathcal{D}(A)$ and $\|Au\| \leqslant a\|u\| + b\|Tu\|, u \in \mathcal{D}(T)$ where a, b are non-negative constants. Then we say that A *is relatively bounded with respect to* T or T*-bounded* and the T*-bound* of A is $\inf b$.

Of course, every bounded operator is T-bounded, for any T, with T-bound equal to zero. Let $\gamma(T)$ denote the reduced minimum modulus of $T \in C(H)$. Then, we know $\mathcal{R}(T)$ is closed if and only if $\gamma(T) > 0$ for every $T \in C(H)$.

Theorem 2.2. *Let $T \in C(H)$ be a semi-Fredholm operator with $\rho(T) \neq \emptyset$. Then the following are equivalent:*

(i) *T has SVEP at 0*
(ii) *$\sigma_a(T)$ does not cluster at 0*
(iii) *$p(T) < \infty$.*

Proof. (i) $\Leftrightarrow$ (iii) This follows from [4, Theorem 15, p.68].

(i) $\Rightarrow$ (ii) Suppose T has SVEP at zero. Since T is semi-Fredholm operator, so is T^n for all $n \in \mathbb{N}$. Then $\mathcal{R}(T^n)$ is closed for all n, so that $T^\infty(H) = \bigcap_{n=1}^{\infty} \mathcal{R}(T^n)$ is closed.

Since T is semi-Fredholm, so that $\mathcal{R}(T)$ is closed, there exists an $\epsilon > 0$ such that $\gamma(T) > \epsilon$. Consider λ in $0 < |\lambda| < \epsilon$. Then $\|\lambda I x\| = |\lambda|\|x\| < \epsilon\|x\| < \gamma(T)\|x\|$, for all $x \in H$. By [5, Theorem 5.22, p. 236], $T - \lambda I$ is a closed semi-Fredholm operator, so that $\mathcal{R}(T - \lambda I)$ is closed for all $0 < |\lambda| < \epsilon$. Thus we have that if $0 < |\lambda| < \epsilon$, then $\lambda \in \sigma_a(T)$ iff $\lambda \in \sigma_p(T)$.

If $0 \neq x \in \mathcal{N}(T - \lambda I)$ then $x = \frac{1}{\lambda}Tx \in \mathcal{R}(T)$. Also, $T^2x = T(\lambda x) = \lambda Tx = \lambda^2 x$. This implies $x = \frac{1}{\lambda^2}T^2x \in \mathcal{R}(T^2)$. Continuing like this, we get $x \in T^\infty(H)$. Thus, $\mathcal{N}(T - \lambda I) \subseteq T^\infty(H)$ for all $\lambda \neq 0$. This implies that every non-zero eigenvalue of T belongs to $\sigma(T|_{T^\infty(H)})$.

Suppose that 0 is a cluster point of $\sigma_a(T)$. There exists a sequence (λ_n) of non-zero eigenvalues of T such that $\lambda_n \to 0$ as $n \to \infty$. Then $\lambda_n \in \sigma(T|_{T^\infty(H)})$ so that $0 \in \sigma(T|_{T^\infty(H)})$, as the spectrum of an operator is closed. Since T is semi-Fredholm operator, either $\alpha(T)$ or $\beta(T)$ is finite and $\mathcal{R}(T)$ is closed. Then $T|_{T^\infty(H)}$ is onto. Also since T has SVEP at 0, from [4, Corollary 3, p. 62], we have that $T|_{T^\infty(H)}$ is injective so that $0 \notin \sigma(T|_{T^\infty(H)})$, which is a contradiction. Therefore, $\sigma_a(T)$ does not cluster at 0.

(ii) $\Rightarrow$ (i) This holds for all closed linear operators. □

Throughout the rest of the article, we denote by $\Lambda(H) = \{T \in C(H) : \mathcal{D}(T)$ is dense and T is an operator with compact resolvent$\}$.

If T and S (both bounded) satisfy Weyl's theorem, it does not necessarily follow that the orthogonal direct sum $T \oplus S$ satisfies Weyl's Theorem (for example refer [3]). The case is similar for a-Weyl's Theorem. W.Y. Lee [7] and B.P. Duggal & C.S. Kubrusly [3] gave sufficient conditions on bounded operators T and S to ensure that $T \oplus S$ satisfies Weyl's Theorem. In this section, we generalize these results to unbounded operators and give sufficient conditions on $S \in C(K)$, when $T \in \Lambda(H)$, to ensure that $T \oplus S$ satisfies Weyl's and a-Weyl's Theorem.

We begin by giving an example where $T \in \Lambda(H)$ and densely defined operator $S \in C(K)$ is such that it satisfies Weyl's theorem, however their orthogonal direct sum $T \oplus S$ does not satisfy Weyl's theorem.

Example 2.3. Let $H = K = \ell^2$ and let T be the operator defined as

$$T(x_1, x_2, x_3, \dots) = (0, x_2, 2x_3, 3x_4, 4x_5, \dots).$$

Then T is an unbounded operator with compact resolvent. We have $\sigma(T) = \sigma_a(T) = \{0, 1, 2, 3, 4 \dots\} = E_o(T)$. Also, $\sigma_W(T) = \emptyset$. Consider $S_1 \in C(K)$ defined as $S_1(x_1, x_2, x_3, \dots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \dots)$. Then $\sigma(S_1) = \{0\}$ and $\sigma_p(S_1) = \emptyset = E_o(S_1)$. Also, $\alpha(S_1) = 0 \neq \beta(S_1)$, so that $\sigma_W(S_1) = \{0\}$. Further, let $S_2 \in C(K)$ be defined as $S_2(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3, 4x_4, \dots)$. Then $\sigma(S_2) = \{1, 2, 3, \dots\}$ and $\sigma_W(S_2) = \emptyset$. Let $S = S_1 \oplus S_2$. Then $\sigma(S) = \{0, 1, 2, 3, \dots\}$, $\sigma_W(S) = \{0\}$ and $E_o(S) = \{1, 2, 3, \dots\}$ so that S satisfies Weyl's theorem.

Let $R = T \oplus S$ defined on $\mathcal{D}(T) \oplus \mathcal{D}(S) \subseteq H \oplus K = \ell^2 \oplus \ell^2$. Then, $\sigma(R) = \{0, 1, 2, 3, 4, \dots\} = E_o(R)$. However, since $\alpha(R) \neq \beta(R)$, $\sigma_W(R) = \{0\}$. Hence, $0 \notin \sigma(R) \setminus \sigma_W(R)$ but $0 \in E_o(R)$, that is, $R = T \oplus S$ does not satisfy Weyl's theorem even though both its summands do so.

Lemma 2.4. *For every $T \in \Lambda(H)$, $\sigma_W(T) = \emptyset$.*

Proof. Since $T \in \Lambda(H)$, $\sigma(T) = E_o(T)$ and $R_\lambda = (T - \lambda I)^{-1}$ is compact for every $\lambda \in \rho(T)$.

Suppose $\lambda_o \in \sigma_W(T) \subseteq \sigma(T) = E_o(T)$. Then $\lambda_o \in \operatorname{iso} \sigma(T)$ so that there exists $\epsilon > 0$ such that whenever $0 < |\lambda - \lambda_o| < \epsilon$, $\lambda \in \rho(T)$ and hence R_λ is compact. Consider

$$
\begin{aligned}
(T - \lambda_o I)R_\lambda &= (T - \lambda_o I)(T - \lambda I)^{-1} \\
&= (T - \lambda I + \lambda I - \lambda_o I)(T - \lambda I)^{-1} \\
&= I - (\lambda_o - \lambda)(T - \lambda I)^{-1} \\
&= I - K_1 \quad \text{on } H \text{, and}
\end{aligned}
$$

$$(T - \lambda I)^{-1}(T - \lambda_o I) = I - K_2, \text{ on } \mathcal{D}(T)$$

where K_1 and K_2 are compact operators. By [8, Theorem 7.2, p. 157], $(T - \lambda_o I)$ is a Fredholm operator with $\operatorname{ind}(T - \lambda_o I) + \operatorname{ind}\{(T - \lambda I)^{-1}\} = 0$. Therefore, $\operatorname{ind}(T - \lambda_o I) = 0$ and $\lambda_o \notin \sigma_W(T)$, which is a contradiction. Hence $\sigma_W(T) = \emptyset$. □

Remark 2.5. For $T \in \Lambda(H)$, since $\sigma_W(T) = \emptyset$, it follows that $\operatorname{ind}(T - \lambda I) = 0$ for every $\lambda \in \mathbb{C}$.

Theorem 2.6. *Let $T \in \Lambda(H)$. Then, T satisfies:*

(i) $\sigma(T) \setminus \sigma_W(T) = E_o(T)$
(ii) $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_o^a(T)$.

Proof. Since $T \in \Lambda(H)$, $\sigma(T) = \sigma_a(T) = E_o(T) = E_o^a(T)$. Also, for any closed linear operator, we have that $\sigma_{SF_+^-}(T) \subseteq \sigma_W(T)$. The proof now follows from the previous Lemma. □

Q. Bai *et al* [1] proved that if T is a closed upper triangular operator matrix

$$T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A and D are densely defined closed operators and B is a closable operator, then $\sigma_W(A) \cup \sigma_W(D) = \sigma_W(T)$ if and only if

$$
\begin{aligned}
\sigma_{p+}(D) \cap \sigma_{p+}(A^*)^* \subseteq \sigma_W(T), \\
\sigma_{p+}(A) \cap \sigma_{p+}(D^*)^* \subseteq \sigma_W(T)
\end{aligned}
$$

where $\sigma_{p+}(.) = \{\lambda \in \sigma_p(.) : \alpha(. - \lambda I) > \beta(. - \lambda I)\}$. and $\sigma_{p+}(.)^* = \{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma_{p+}(.)\}$.

In particular, if $\sigma_{p+}(D) \cap \sigma_{p+}(A^*)^* = \emptyset$ and $\sigma_{p+}(A) \cap \sigma_{p+}(D^*)^* = \emptyset$, then $\sigma_W(A) \cup \sigma_W(D) = \sigma_W(T)$ holds.

We now extend the result of W.Y. Lee [7] for Weyl's theorem for direct sums of bounded linear operators to the case when the summands are not necessarily bounded.

Theorem 2.7. *If $T \in \Lambda(H)$ and $S \in C(K)$ is a densely defined isoloid operator satisfying Weyl's theorem, then $T \oplus S$ satisfies Weyl's theorem.*

Proof. Since $T \in \Lambda(H)$, $\operatorname{ind}(T-\lambda I) = 0$ for every $\lambda \in \mathbb{C}$. Hence, $\sigma_{p+}(T) = \sigma_{p+}(T^*)^* = \emptyset$ and $\sigma_W(T) \cup \sigma_W(S) = \sigma_W(T \oplus S)$.

Clearly, T is isoloid and S is isoloid by hypothesis. Using that $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$ for any two closed operators S and T, we have that

$$E_o(T \oplus S) = \{E_o(T) \cap \rho(S)\} \cup \{\rho(T) \cap E_o(S)\} \cup \{E_o(T) \cap E_o(S)\}.$$

By hypothesis, S satisfies Weyl's theorem and by Theorem 2.6(i), T satisfies Weyl's theorem, thus we have

$$\begin{aligned}
&\sigma(T \oplus S) \setminus \sigma_W(T \oplus S) \\
&\quad = \{\sigma(T) \cup \sigma(S)\} \setminus \{\sigma_W(T) \cup \sigma_W(S)\} \\
&\quad = \{(\sigma(T) \setminus \sigma_W(T)) \cap \rho(S)\} \cup \{\rho(T) \cap (\sigma(S) \setminus \sigma_W(S))\} \cup \\
&\qquad\qquad \{(\sigma(T) \setminus \sigma_W(T)) \cap (\sigma(S) \setminus \sigma_W(S))\} \\
&\quad = \{E_o(T) \cap \rho(S)\} \cup \{\rho(T) \cap E_o(S)\} \cup \{E_o(T) \cap E_o(S)\} \\
&\quad = E_o(T \oplus S).
\end{aligned}$$

Hence, $T \oplus S$ satisfies Weyl's theorem. □

Remark 2.8. Example 2.3 shows that the condition that $S \in C(K)$ is "isoloid", in the above theorem, cannot be dropped.

The following corollary is an immediate consequence of the above theorem.

Corollary 2.9. *Let $T \in \Lambda(H)$ and suppose $S \in C(K)$ is densely defined and satisfies Weyl's theorem such that $\operatorname{iso}\sigma(S) = \emptyset$. Then, $T \oplus S$ satisfies Weyl's theorem.*

The following theorem shows that the converse of the above theorem holds true.

Theorem 2.10. *Let $T \in \Lambda(H)$ and $S \in C(K)$ be a densely defined isoloid operator. If $T \oplus S$ satisfies Weyl's theorem, then so does S.*

Proof. Since $T \in \Lambda(H)$, $\sigma_W(T) \cup \sigma_W(S) = \sigma_W(T \oplus S)$ for every densely defined operator $S \in C(K)$. Also,

$$E_o(T \oplus S) = \{E_o(T) \cap \rho(S)\} \cup \{\rho(T) \cap E_o(S)\} \cup \{E_o(T) \cap E_o(S)\}$$

since T and S both are isoloid. We need to show $\sigma(S) \setminus \sigma_W(S) = E_o(S)$.

Suppose $\lambda \in \sigma(S) \setminus \sigma_W(S)$. Since $\sigma_W(T) = \emptyset$, $T - \lambda I$ and $S - \lambda I$ both are Weyl and thus, $(T \oplus S) - \lambda(I \oplus I)$ is Weyl. Now,

$$\begin{aligned}
\lambda &\in \sigma(T \oplus S) \setminus \sigma_W(T \oplus S) \\
&= E_o(T \oplus S) \\
&= \{E_o(T) \cap \rho(S)\} \cup \{\rho(T) \cap E_o(S)\} \cup \{E_o(T) \cap E_o(S)\} \\
&\subseteq \{E_o(T) \cap \rho(S)\} \cup E_o(S).
\end{aligned}$$

As $\lambda \in \sigma(S)$, we have $\lambda \in E_o(S)$.

Conversely, suppose $\lambda \in E_o(S)$. Then $\lambda \in \sigma_p(T \oplus S)$ and $\alpha(T \oplus S - \lambda(I \oplus I)) = \alpha(T - \lambda I) + \alpha(S - \lambda I) < \infty$. We shall show that $\lambda \in \operatorname{iso}\sigma(T \oplus S)$. We know that,

$$\operatorname{iso}\sigma(T \oplus S) = [\operatorname{iso}\sigma(T) \cup \operatorname{iso}\sigma(S)] \setminus [\{\operatorname{iso}\sigma(T) \cap \operatorname{acc}\sigma(S)\} \cup \{\operatorname{iso}\sigma(S) \cap \operatorname{acc}\sigma(T)\}].$$

Since λ is isolated in $\sigma(S)$ and $\operatorname{acc}\sigma(T) = \emptyset$, we have that $\lambda \in \operatorname{iso}\sigma(T \oplus S)$. Therefore, $\lambda \in E_o(T \oplus S) = \sigma(T \oplus S) \setminus \sigma_W(T \oplus S)$. Thus, $\operatorname{ind}(S - \lambda I) = \operatorname{ind}(T - \lambda I) + \operatorname{ind}(S - \lambda I) = \operatorname{ind}(T \oplus S - \lambda(I \oplus I)) = 0$ and $\lambda \in \sigma(S) \setminus \sigma_W(S)$. □

Theorem 2.11. *Let $T \in \Lambda(H)$ and $S \in C(K)$ be a densely defined a-isoloid operator. Then S satisfies a-Weyl's theorem if and only if $T \oplus S$ satisfies a-Weyl's theorem.*

Proof. By theorem 2.6(ii), $\sigma_{SF_+^-}(T) = \emptyset$ and T satisfies a-Weyl's theorem. Then $\sigma_{SF_+^-}(T \oplus S) \subseteq \sigma_{SF_+^-}(T) \cup \sigma_{SF_+^-}(S) = \sigma_{SF_+^-}(S)$.

If $\lambda \notin \sigma_{SF_+^-}(T \oplus S)$, then $\alpha(T \oplus S - \lambda(I \oplus I)) < \infty$, $\mathcal{R}(T \oplus S - \lambda(I \oplus I))$ is closed and $\operatorname{ind}(T \oplus S - \lambda(I \oplus I)) \leqslant 0$. Thus we get, $\alpha(S - \lambda I) \leqslant \alpha(T - \lambda I) + \alpha(S - \lambda I) = \alpha(T \oplus S - \lambda(I \oplus I)) < \infty$, $\mathcal{R}(S - \lambda I)$ is closed and $\operatorname{ind}(S - \lambda I) = \operatorname{ind}(T - \lambda I) + \operatorname{ind}(S - \lambda I) = \operatorname{ind}(T \oplus S - \lambda(I \oplus I)) \leqslant 0$. Therefore, $\lambda \notin \sigma_{SF_+^-}(S)$ so that $\sigma_{SF_+^-}(S) \subseteq \sigma_{SF_+^-}(T \oplus S)$. Hence, $\sigma_{SF_+^-}(T \oplus S) = \sigma_{SF_+^-}(S) = \sigma_{SF_+^-}(T) \cup \sigma_{SF_+^-}(S)$.

Suppose S satisfies a-Weyl's theorem. Then, since T and S both satisfy a-Weyl's theorem, we have

$$\begin{aligned}
&\sigma_a(T \oplus S) \setminus \sigma_{SF_+^-}(T \oplus S) \\
&\qquad = \{\sigma_a(T) \cup \sigma_a(S)\} \setminus \{\sigma_{SF_+^-}(T) \cup \sigma_{SF_+^-}(S)\} \\
&\qquad = \{E_o^a(T) \cap \rho_a(S)\} \cup \{\rho_a(T) \cap E_o^a(S)\} \cup \{E_o^a(T) \cap E_o^a(S)\} \\
&\qquad = E_o^a(T \oplus S) \qquad (\because \text{ S and T are a-isoloid}).
\end{aligned}$$

Hence, $T \oplus S$ satisfies a-Weyl's theorem.

Suppose now that $T \oplus S$ satisfies a-Weyl's theorem and let $\lambda \in \sigma_a(S) \setminus \sigma_{SF_+^-}(S)$. Then by [1, Lemma 2.2(ii)], we have

$$\begin{aligned}
&\lambda \in \sigma_a(T \oplus S) \setminus \sigma_{SF_+^-}(T \oplus S) = E_o^a(T \oplus S) \\
&\quad = \{E_o^a(T) \cap \rho_a(S)\} \cup \{\rho_a(T) \cap E_o^a(S)\} \cup \{E_o^a(T) \cap E_o^a(S)\} \\
&\quad \subseteq \{E_o^a(T) \cap \rho_a(S)\} \cup E_o^a(S).
\end{aligned}$$

Since $\lambda \in \sigma_a(S)$, we have $\lambda \in E_o^a(S)$. Conversely, suppose $\lambda \in E_o^a(S)$. Using similar argument as in Theorem 2.10, we have $\lambda \in \sigma_a(T \oplus S)$, $\alpha(T \oplus S - \lambda(I \oplus I)) < \infty$ and $\lambda \in \operatorname{iso}\sigma_a(T \oplus S)$. Thus, $\lambda \in E_o^a(T \oplus S)$. Then $\lambda \notin \sigma_{SF_+^-}(T \oplus S) = \sigma_{SF_+^-}(S)$. Hence, S satisfies a-Weyl's theorem. □

As an immediate consequence of the above theorem, we have the following corollary:

Corollary 2.12. *Let $T \in \Lambda(H)$ and $S \in C(K)$ be a densely defined operator with $\operatorname{iso}\sigma_a(S) = \emptyset$. Then S satisfies a-Weyl's theorem if and only if $T \oplus S$ satisfies a-Weyl's theorem.*

3. Browder's Theorem for Direct sums of operators

In this section, we study the Browder's theorem for direct sums of unbounded operators. It is shown that if $T \in \Lambda(H)$ and densely defined operator $S \in C(K)$ satisfies Browder's Theorem then $T \oplus S$ satisfies Browder's Theorem. The converse also holds true. Another sufficient condition for $T \oplus S$ to satisfy the Browder's theorem, when $T \in \Lambda(H)$, is that S has SVEP.

The following theorem proves the Browder and a-Browder's Theorem for operators with compact resolvent.

Theorem 3.1. *If $T \in \Lambda(H)$, then T satisfies:*
(i) $\sigma(T) \setminus \sigma_W(T) = \pi_o(T)$
(ii) $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$.

Proof. (i) Since $\sigma_W(T) = \emptyset$, $T - \lambda I$ is Weyl for every $\lambda \in \sigma(T)$ so that $\alpha(T - \lambda I) = \beta(T - \lambda I) < \infty$. Also since every $\lambda \in \sigma(T) = E_o(T)$ is isolated, we have $\sigma_a(T)$ does not cluster at λ. By Theorem 2.2, we have $p(T - \lambda I) < \infty$, so that by [9, Theorem 4.5(c)], $q(T - \lambda I) = p(T - \lambda I) < \infty$. Thus, $\lambda \in \pi_o(T)$ and therefore $\sigma(T) \subseteq \pi_o(T)$. Since the reverse inclusion holds for all operators, we have $\sigma(T) \setminus \sigma_W(T) = \sigma(T) = \pi_o(T)$. Hence, T satisfies Browder's theorem.

(ii) Suppose $\lambda \in \sigma_a(T)$. By Lemma 2.4, $\sigma_W(T) = \emptyset$ so that $T - \lambda I$ is Fredholm. Also every $\lambda \in \sigma(T)$ is isolated, so that by Theorem 2.2, $p(T - \lambda I) < \infty$. Thus, $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T)$ and $\sigma_{ub}(T) \subseteq \sigma_{SF_+^-}(T)$. Since the reverse inclusion holds for all operators, we get $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$. Hence, T satisfies a-Browder's theorem.

□

Remark 3.2. In the proof of part (i) above we have implicitly proved that $\sigma_b(T) = \emptyset$ when $T \in \Lambda(H)$.

For any closed densely defined linear operator T with $\rho(T) \neq \emptyset$, $\sigma(T) \setminus \sigma_b(T) = \pi_o(T)$ (see [6], [9]). Also, in [1] it is proved that if T is a closed upper triangular operator matrix

$$T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A and D are densely defined closed operators and B is a closable operator, then $\sigma_b(A) \cup \sigma_b(D) = \sigma_b(T) \cup \sigma_{asc}(D)$ where $\sigma_{asc}(.) = \{\lambda \in \mathbb{C} : p(. - \lambda I) = \infty\}$.

We shall use these results to give some necessary conditions on S, when $T \in \Lambda(H)$, which ensure that $T \oplus S$ satisfies Browder's Theorem.

Theorem 3.3. *Suppose $T \in \Lambda(H)$ and $S \in C(K)$ has dense domain and satisfies Browder's theorem. Then, $T \oplus S$ satisfies Browder's theorem.*

Proof. If $T \in \Lambda(H)$, we have shown that $\sigma_W(T \oplus S) = \sigma_W(T) \cup \sigma_W(S)$. Further, since $\sigma_b(T) = \emptyset$, it is easy to prove that $\sigma_{asc}(S) \subseteq \sigma_b(T \oplus S)$ so that $\sigma_b(T \oplus S) =$

$\sigma_b(T) \cup \sigma_b(S)$. Therefore, we have

$$\begin{aligned}
\pi_o(T \oplus S) &= \sigma(T \oplus S) \setminus \sigma_b(T \oplus S) \\
&= \{\sigma(T) \cup \sigma(S)\} \setminus \{\sigma_b(T) \cup \sigma_b(S)\} \\
&= \{(\sigma(T) \setminus \sigma_b(T)) \cap \rho(S)\} \cup \{\rho(T) \cap (\sigma(S) \setminus \sigma_b(S))\} \cup \\
&\qquad\qquad \{(\sigma(T) \setminus \sigma_b(T)) \cap (\sigma(S) \setminus \sigma_b(S))\} \\
&= \{\pi_o(T) \cap \rho(S)\} \cup \{\rho(T) \cap \pi_o(S)\} \cup \{\pi_o(T) \cap \pi_o(S)\} \\
&= \{(\sigma(T) \setminus \sigma_W(T)) \cap \rho(S)\} \cup \{\rho(T) \cap (\sigma(S) \setminus \sigma_W(S))\} \cup \\
&\qquad\qquad \{(\sigma(T) \setminus \sigma_W(T)) \cap (\sigma(S) \setminus \sigma_W(S))\} \\
&= \{\sigma(T) \cup \sigma(S)\} \setminus \{\sigma_W(T) \cup \sigma_W(S)\} \\
&= \sigma(T \oplus S) \setminus \sigma_W(T \oplus S)
\end{aligned}$$

where the third last equality follows from the fact that T satisfies Browder's theorem by Theorem 3.1(i) and S satisfies Browder's theorem by hypothesis.

Hence, $T \oplus S$ satisfies Browder's theorem. □

Since every closed densely defined operator with SVEP satisfies Browder's theorem, the following corollary is an immediate consequence of the above theorem:

Corollary 3.4. *Suppose $T \in \Lambda(H)$ and $S \in C(K)$ with dense domain has SVEP. Then, $T \oplus S$ satisfies Browder's theorem.*

We shall now prove the converse of Theorem 3.3:

Theorem 3.5. *Suppose $T \in \Lambda(H)$ and $S \in C(K)$ has dense domain. Then, S satisfies Browder's theorem if $T \oplus S$ satisfies Browder's theorem.*

Proof. Since, $\sigma(S) \setminus \sigma_b(S) = \pi_o(S)$, we need to show that $\sigma_b(S) = \sigma_W(S)$. It is enough to show $\sigma_b(S) \subseteq \sigma_W(S)$ since the reverse inclusion holds true for every operator. Suppose $\lambda \notin \sigma_W(S)$, then $S - \lambda I$ is Weyl and hence $T \oplus S$ is Weyl. Now $T \oplus S$ satisfies Browder's theorem, so $\lambda \in \pi_o(T \oplus S)$. Then $p(S - \lambda I) < \infty$ and $q(S - \lambda I) < \infty$, hence $\lambda \notin \sigma_b(S)$. □

The following example illustrates the results proved in this article.

Example 3.6. Let $H = K = \ell^2$ and let T be the operator defined as

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3, 4x_4, 5x_5, \dots).$$

Then T is an unbounded operator with compact resolvent. We have $\sigma(T) = \sigma_a(T) = \{1, 2, 3, 4 \dots\} = E_o(T) = E_o^a(T) = \pi_o(T) = \pi_o^a(T)$. Also, $\sigma_W(T) = \sigma_{SF_+^-}(T) = \sigma_{ub}(T) = \emptyset$. Consider $S \in C(K)$ to be defined as

$$S(x_1, x_2, x_3, \dots) = (0, x_1, 2x_2, 3x_3, \dots).$$

Then $\sigma(S) = \mathbb{C}$, $\sigma_a(S) = \emptyset$ and $\sigma_p(S) = \emptyset = E_o(S) = E_o^a(S) = \pi_o(S)$. For any arbitrary $\lambda \in \mathbb{C}$, $\alpha(S - \lambda I) = 0 \neq \beta(S - \lambda I)$ and $\mathcal{R}(S - \lambda I)$ is closed. Thus $\sigma_W(S) = \sigma_b(S) = \mathbb{C}$ and $\sigma_{SF_+^-}(S) = \emptyset$. Hence, T and S satisfy Weyl's theorem, a-Weyl's theorem and Browder's theorem. Also, S is both isoloid and a-isoloid.

Now let $R = T \oplus S$ on the Hilbert space $\ell^2 \oplus \ell^2$. Then, $\sigma(R) = \mathbb{C}$, $\sigma_a(R) = \{1, 2, 3, 4, \dots\} = E_o^a(R)$ and since $\operatorname{iso} \sigma(R) = \emptyset$, $E_o(R) = \pi_o(R) = \emptyset$. Also,

$\sigma_W(R) = \sigma_b(R) = \mathbb{C}$ and $\sigma_{SF_+^-}(R) = \emptyset$. Hence, R satisfies Weyl's theorem, a-Weyl's theorem and Browder's theorem.

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