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PARTIAL ISOMETRIES: A SURVEY

FRANCISCO J. FERNÁNDEZ-POLO,[1] and ANTONIO M. PERALTA[2*]

This paper is dedicated to the memory of Professor Uffe Haagerup

ABSTRACT. We survey the main results characterizing partial isometries in C*-algebras and tripotents in JB*-triples obtained in terms of regularity, conorm, quadratic-conorm, and the geometric structure of the underlying Banach spaces.

1. INTRODUCTION

It is known that, even in the most favorable case of two by two matrices with complex entries, an arbitrary matrix cannot be always expressed as a complex linear combination of mutually orthogonal projections (i.e. hermitian idempotents). If we relax our requirements and we replace projections with partial isometries, then every square matrix can be written as a linear combination of mutually orthogonal partial isometries with positive coefficients. This is just one of our favorite motivations to introduce the potential readers into the notion of partial isometries, or in a more general setting tripotents. Partial isometries have been intensively studied since the very early stages of the theory of linear operator on complex Hilbert spaces (see, for example, [18, 29]). These objects play a fundamental role in operator theory, C*-algebras and other generalizations in more general Jordan-Banach structures like JB*-algebras and JB*-triples.

But what is a partial isometry? Most of basic references and books place the origins of partial isometries in the space $B(H)$ of all bounded linear operators on

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*Corresponding author.
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a complex Hilbert space H. An element e in $B(H)$ is a *partial isometry* if it acts isometrically on the orthogonal complement of its kernel, that is, $\|e(\xi)\| = \|\xi\|$ for every $\xi \in (\ker(e))^{\perp} = \{\xi \in H : \langle \xi | \eta \rangle = 0, \ \forall \eta \in \ker(e)\}$ (see [29, §127], [18, Definition 3.8]). Applying the completeness of H and the isometric behavior of e on $(\ker(e))^{\perp}$ we can easily see that $\operatorname{ran}(e) = e(H) = e((\ker(e))^{\perp})$ is a closed subspace of H. Following the standard notation, the space $(\ker(e))^{\perp}$ is called the *initial space of* e, while $\operatorname{ran}(e)$ is called the *final space of* e.

One of the earliest results in operator theory (see, for example, [29, Problem 127 and subsequent Corollaries], or [55, §2.2.8] or [18, Exercises 9 and 10 in page 16]) establishes any of the following statements are equivalent for an element e in $B(H)$:

(a) e is a partial isometry;
(b) ee^* is an idempotent (actually, ee^* is the orthogonal projection of H onto the final space of e);
(c) e^*e is an idempotent (actually, e^*e is the orthogonal projection of H onto the initial space of e);
(d) $ee^*e = e$;
(e) e^* is a partial isometry,

where e^* denotes the adjoint of e in $B(H)$.

The above equivalences are just the beginning of a long series of results seeking for characterizations of partial isometries in several structures generalizing the algebraic-geometric structure of $B(H)$ in different directions.

An element e in a general C*-algebra A is said to be a *partial isometry* if $ee^*e = e$. It is easy to check that, in this case, ee^* and e^*e are projections (i.e. self-adjoint idempotents) in A. On the other hand, if ee^* is a projection in A, it can be easily checked that

$$(e - ee^*e)(e - ee^*e)^* = (e - ee^*e)(e^* - e^*ee^*)$$
$$= ee^* - ee^*ee^* - ee^*ee^* + ee^*ee^*ee^* = 0,$$

which implies, via Gelfand–Naimark axiom that $e = ee^*e$. Similarly, e^*e being a projection assures that $e^*ee^* = e^*$ (or equivalently, $ee^*e = e$).

The aim of this paper is to survey some of the most meaningful (and useful) characterizations of partial isometries in real and complex C*-algebras, ternary ring of operators and real and complex JB*-triples obtained, by different authors, during the last forty years. For completeness reasons, and in order to offer an unified approach, we have tried to modify and gather together some of the results dispersed in the literature contributing, in most of cases, with an alternative approach to the original statements and proofs. So, contrary to an orthodox survey, we will not be limited to list results and references.

We begin section 2 with a revision of the notion of von Neumann regularity and Moore–Penrose invertibility in the setting of C*-algebras. In a non-unital Banach algebra the notion of invertibility makes no sense. However, there exist non-unital Banach algebras (respectively, C*-algebras) containing a wide set of idempotents (respectively, projections). The notion of von Neumann regularity is more appropriate in this setting. We recall that an element a in an associative Banach algebra A is called *von Neumann regular* or simply *regular* if there exists $b \in A$ satisfying $aba = a$. The element b is one of the many different choices satisfying the previous identity, it is called a *generalized inverse* of a. If b and a are generalized inverses of each other, we say that b is a normalized generalized inverse of a.

The order provided by the cone of positive elements in a C*-algebra is the key ingredient to define Moore–Penrose invertibility. An element a in a C*-algebra A is *Moore–Penrose invertible* if there exists b in A such that $aba = a$, $bab = b$ and ab, ba are projections (i.e. self adjoint idempotents) in A. The element b in the definition of Moore–Penrose invertibility is unique, it is called the Moore–Penrose inverse of a, and it is denoted by $a^\dagger$. In Theorem 2.3 we present the main equivalent reformulations of regularity in the setting of C*-algebras derived from studies due to R. Harte and M. Mbekhta [34] and M. Mbekhta [51]. Briefly speaking, the following statements are equivalent for every element a in a C*-algebra A.

- $(b.1)$ a is regular;
- $(b.2)$ 0 is an isolated point in $\sigma(|a|) \cup \{0\}$, where $|a| = (a^*a)^{\frac{1}{2}}$;
- $(b.3)$ The partial isometry appearing in the polar decomposition of a lies in A and $|a|$ is invertible in the C*-subalgebra generated by $|a|$;
- $(b.4)$ The hereditary C*-subalgebra generated by $|a|$ is unital and $|a|$ is invertible in this C*-subalgebra;
- $(b.5)$ a is Moore–Penrose invertible.

The proofs here differ from the originals because in this note the arguments are built around the polar decomposition of an element in a C*-algebra.

Our list of results characterizing partial isometries begins with Theorem 2.6 ([4, Theorem 2.1]), where it is shown that for a norm-one element e in a C*-algebra A, the following statements are equivalent:

- (a) e is a partial isometry;
- (b) e is regular and $\|e^\dagger\| \leq 1$;
- (c) e is regular and admits a generalized inverse b with $\|b\| \leq 1$.

The second characterization of partial isometries is given in terms of the conorm of an element in a C*-algebra A. The notion of conorm was introduced by R. Harte and M. Mbekhta [35] in terms of the reduced minimum modulus of the left and right multiplication operators. We recall that the *reduced minimum modulus* of a non-zero bounded linear operator T between Banach spaces X and Y, is defined by

$$\gamma(T) := \inf\{\|T(x)\| : \mathrm{dist}(x, \ker(T)) \geq 1\},$$

and $\gamma(0) = 0$ (see page 86 for more details). If for each a in a Banach algebra A, L_a (respectively, R_a) denotes the left (respectively, right) multiplication operator by the element a, the *left conorm* (respectively, *right conorm*) of a in A is the quantifier

$$\gamma^l(a) = \gamma(L_a) \quad \text{(respectively, } \gamma^r(a) = \gamma(R_a)\text{)}.$$

In Theorem 2.9 we revisit the main conclusions about conorm in [35]. Concretely, for an element a in a C*-algebra A the following statements hold:
(a) $\gamma^l(a)^2 = \gamma^l(|a|)^2 = \gamma^l(|a|^2) = \inf\{t : t \in \sigma(a^*a)\backslash\{0\}\}$;
(b) $\gamma^r(a)^2 = \gamma^r(|a|)^2 = \gamma^r(|a|^2) = \inf\{t : t \in \sigma(a^*a)\backslash\{0\}\}$;
(c) $\gamma^l(a)^2 = \gamma^l(|a|)^2 = \gamma^l(|a|^2) = \gamma^r(a)^2 = \gamma^r(|a|)^2 = \gamma^r(|a|^2)$;
(d) $\gamma^l(a) = \gamma^l(a^*) = \gamma^r(a) = \gamma^r(a^*) \leq \|a\|$;
(e) When a is regular $\gamma^l(a) = \gamma^l(a^*) = \gamma^r(a) = \gamma^r(a^*) = \frac{1}{\|a^\dagger\|}$.

Thanks to the above result, the *conorm* on a C*-algebra A is the mapping $\gamma : A \to \mathbb{R}_0^+$ defined by

$$\gamma(a) := \gamma^l(a) = \gamma^r(a) \ (\forall a \in A).$$

Theorem 2.10 contains a characterization of non-zero partial isometries in terms of their conorm, which is also due to Harte and Mbekhta (see [35, (4.9)]). The concrete result reads as follows: Let e be a norm-one element in a C*-algebra A. Then the following statements are equivalent:

(a) e is a partial isometry;
(b) $\gamma(e) = 1$ (equivalently, $\gamma(e) \geq 1$).

Subsection 2.1 is dedicated to the study of regular elements and conorm in the setting of real C*-algebras. The characterizations of partial isometries in terms of Moore–Penrose invertibility and in terms of the conorm are extended to the setting of real C*-algebras in Theorems 2.13 and 2.14.

Section 3 is devoted to survey a groundbreaking result due to C.A. Akemann and N. Weaver (see [2]), which characterizes partial isometries of a C*-algebra A in terms the geometric Banach space structure of A. In this section we present a unified approach to deal with real and complex C*-algebras. The characterization enjoys the additional virtue of having a simple formulation. For each norm-one element x in a real or complex Banach space X we consider the following sets:

$$D_1^X(x) := \Big\{ y \in X : \text{ there exists } \alpha > 0 \text{ with } \|x \pm \alpha y\| = 1 \Big\},$$

$$D_2^X(x) := \{ y \in X : \|x + \beta y\| = \max\{1, \|\beta y\|\}, \text{ for all } \beta \in \mathbb{R}\}.$$

Let us observe that $D_1^X(x)$ and $D_2^X(x)$ are just determined by the geometric structure of the Banach space X. The commented characterization asserts that a norm-one element e in a real or complex C*-algebra A, is a partial isometry if, and only if, $D_1^A(e) = D_2^A(e)$ (see Theorem 3.2, [2, Theorem 1]).

The geometric characterization of partial isometries can be applied to rediscover a celebrated result of R.V. Kadison characterizing the extreme points of the closed unit ball of a C*-algebra (see Corollary 3.4).

As it was observed during the seventies, eighties, and nineties by authors like L.A. Harris [32], M. Koecher [46], A. Fernández López, E. García Rus, E. Sánchez Campos, and M. Siles Molina [23], and W. Kaup [44], the notion of regularity can be also considered in the setting of Jordan triple systems and complex JB*-triples (see the first paragraphs of section 4 for the concrete definitions of these mathematical objects). We employ the first part of section 4 to revisit the main results on this topic.

We recall that an element a in a Jordan triple system E is (*von Neumann*) *regular* if a belongs to $Q(a)(E)$ (i.e. there exists b in E such that $Q(a)(b) = a$) and *strongly* (*von Neumann*) *regular* if there exists b in E such that $a = Q(a)^2(b) = \{a, \{a, b, a\}, a\}$.

In Theorem 4.3 (see also Proposition 4.8) we gather together the different characterizations of the notion of regularity in the setting of JB*-triples borrowed from [46], [23], [44] and [15]. Concretely, for an element a in a JB*-triple E the following statements are equivalent:

a) a is regular;
b) 0 is isolated in the triple spectrum of a;
c) $Q(a)$ has closed range;
d) There exists an element b in E satisfying the following properties $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)\,Q(b) - Q(b)\,Q(a) = 0$;
e) a is strongly regular.

For a regular element a, the element b appearing in statement (d) above is unique and is called *the generalized inverse* of a (denoted by $a^\dagger$).

In a JB*-triple an element e is called *tripotent* if $\{e, e, e\} = e$. When a C*-algebra A is regarded as a JB*-triple with respect to the product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$, tripotents and partial isometries in A coincide. Therefore, characterizing tripotents in JB*-triples is a Jordan analogue of characterizing partial isometries in C*-algebras. The notion of regularity can be applied to get a first characterization of tripotents; namely, a norm-one element e in a JB*-triple E is a tripotent if, and only if, it is regular and $\|e^\dagger\| \leq 1$ if, and only if, there exists b in E with $\|b\| \leq 1$ and $Q(e)(b) = e$ (see Theorem 4.7, [15, Corollary 3.6]).

The quadratic-conorm in the setting of JB*-triples was introduced by M.J. Burgos, A. Kaidi, A. Morales, M.I. Ramírez and the second author of this note in [15]. For an element a in a JB*-triple E, its *quadratic-conorm*, $\gamma^q(a)$, is defined as the reduced minimum modulus of the conjugate-linear operator $Q(a)$, that is, $\gamma^q(a) = \gamma(Q(a))$. The fundamental property of the quadratic-conorm is revisited in Theorem 4.9 where it is shown that the identity

$$\gamma^q(a) = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\}$$

holds for every non-zero element a in a JB*-triple E. Consequently, $\|a\|^2 \geq \gamma^q(a)$, for all $a \in E$, and

$$\gamma^q(a) = \|a^\dagger\|^{-2}, \text{ whenever } a \text{ is regular.}$$

As in the case of C*-algebras, the quadratic-conorm was applied by M.J. Burgos, A. Kaidi, A. Morales, M.I. Ramírez and the second author of this note to characterize tripotents. The result is presented here in Theorem 4.10, where it is proved that a norm-one element e in a JB*-triple E is a tripotent if, and only if, $\gamma^q(e) = 1$.

We devote subsection 4.1 to explore different studies on regularity and quadratic-conorm in the setting of real JB*-triples.

In section 5 we deal with the geometric characterization of tripotents in real and complex JB*-triples obtained by J. Martínez and the authors of this survey in [24]. By generalizing the geometric characterization of partial isometries due to Akemann and Weaver, it can be shown that a norm-one element e in a real or complex JB*-triple E is a tripotent if, and only if, $D_1^E(e) = D_2^E(e)$ (see Theorem 5.2). We apply this characterization to rediscover the classic results by L. Harris in [31, Theorem 11], W. Kaup and H. Upmeier [45, Proposition 3.5] and J.M. Isidro, W. Kaup and A. Rodríguez [37, Lemma 3.3] describing the extreme points of the closed unit ball of J*-algebras, complex JB*-triples and real JB*-triples, respectively.

2. First algebraic approaches: generalized inverses and conorms

Chronologically speaking, the first studies on partial isometries are related to the notion of regular elements and Moore–Penrose inverses with contributions conducted by R. Harte [33], R. Harte and M. Mbekhta [34, 35], M. Mbekhta [51] and C. Badea and M. Mbekhta [6]. These results will be reviewed in this section.

We recall that an element a in an associative ring $\mathcal{R}$ is said to be *regular* or *von Neumann regular* if there exists $b \in \mathcal{R}$ satisfying $aba = a$. The element b is called a *generalized inverse* of a, but it need not be unique. If b is a generalized inverse of a and the equality $bab = b$ holds, we say that b is a *normalized generalized inverse* of a. In a unital ring every invertible element in the usual sense is regular. However, in non-unital rings the notion of regularity provides elements which are "locally" invertible. For example, the algebra c_0 of all null sequences does not contain a unit element and the notion of invertibility does not make any sense. However, it can be easily checked that the set of regular elements in c_0 is precisely the subspace c_{00} of all eventually null sequences.

If b is a generalized inverse of an element a in an associative ring $\mathcal{R}$, then the elements $ab = abab = (ab)(ab)$, and $ba = baba = (ba)(ba)$ both are idempotents and satisfy $a(ba) = (ab)a = a$.

For our particular goals, an element a in a Banach algebra A will be associated with its left and right multiplication operators defined by $L_a, R_a : A \to A$, $x \mapsto L_a(x) = ax$ and $R_a(x) = xa$, respectively. The quadratic mapping $U_a : A \to A$ is given by $U_a(x) := axa$ $(x \in A)$. If we assume that a is regular and b is any generalized inverse of a, then the identities

$$L_a = L_a L_b L_a, \text{ and } R_a = R_a R_b R_a,$$

assure that L_a and R_a are regular elements in $B(A)$ the Banach algebra of all bounded linear operators on A.

Proposition 2.1. [34, Theorem 2] *Let a be a regular element in a Banach algebra A. Then the mappings L_a, R_a and U_a have closed range.*

Proof. Let us take $b \in A$ such that $aba = a$. The maps $P = L_a L_b$ and $Q = L_b L_a$ are idempotents in $B(A)$ with $PL_a = L_a Q = L_a$. Suppose $L_a(x_n) \to z$ in norm with $z \in A$, $(x_n) \subset A$. Then $Q(x_n) = L_b L_a(x_n) \to bz \in A$ and $L_a Q(x_n) \to a(bz)$. Since $L_a Q(x_n) = L_a(x_n) \to z$, we deduce that $a(bz) = z \in L_a(A)$. The statement concerning R_a follows by similar arguments.

Suppose now that $U_a(x_n) \to z$ in norm for a suitable sequence (x_n) in A. Clearly, $U_bU_a(x_n) \to U_b(z)$ in norm. Since $U_aU_bU_a = U_{aba} = U_a$, we arrive at $U_a(x_n) = U_aU_bU_a(x_n) \to U_aU_b(z)$, which assures that $U_aU_b(z) = z \in U_a(A)$, as desired. □

The existence of regular elements admitting a wide set of generalized inverses makes necessary a more restrictive definition which narrows the set of generalized inverses. The setting of C*-algebras is specially appropriate for this purpose. C*-algebras are widely known by mathematicians and there is no need to revisit the formal definition. The reader interested in more details is referred to the monographs [58, 55] and [59].

Let A be a C*-algebra. An element a in A is *Moore–Penrose invertible* if it admits a normalized generalized inverse b such that ab and ba are projections in A, that is, $(ab)^* = ab$ and $(ba)^* = ba$. In these conditions, we shall say that b is a Moore–Penrose inverse of a.

Every partial isometry e in a C*-algebra A is Moore–Penrose invertible because in this case $e = ee^*e$, $e^* = e^*ee^*$ and ee^* and e^*e are projections in A.

Henceforth, the C*-subalgebra generated by a symmetric element a in a C*-algebra A will be denoted by A_a. The symbols A_{sa} and A^+ will denote the sets of all self-adjoint elements in A and the positive cone of A, respectively.

Continuous functional calculus and polar decompositions are useful tools in the theory of C*-algebras. Following standard notation, given an element a in a C*-algebra A, the symbol $|a|$ will denote the element $(a^*a)^{\frac{1}{2}} \in A$. The spectrum of an element $a \in A$ will be denoted by $\sigma(a)$. It is known that $\sigma(|a|) \cup \{0\} = \sigma(|a^*|) \cup \{0\}$, for every $a \in A$. We shall frequently regard A as a C*-subalgebra of its bidual A^{**}.

For each element a in A^+, the *range projection* (also called support projection in [58, Definition 1.10.3]) of a in A^{**} is the least projection among all projections p in A^{**} such that $ap = pa = a$ and it will be denoted by $r(a)$ (see [55, 2.2.7] or [53, Theorem 4.1.9]). It is also known that the sequence $((\frac{1}{n} + a)^{-1}a) \subset A_a$ is monotone increasing to $r(a)$, and in particular $((\frac{1}{n} + a)^{-1}a) \to r(a)$ in the weak* topology of A^{**} (see [55, 2.2.7]).

Let $a = v|a|$ be the polar decomposition of a in A^{**}, where v is a partial isometry in A^{**}, which, in general, does not belong to A (compare [58, Theorem 1.12.1] or [55, Proposition 2.2.9]). It is further known that v is the unique partial isometry in A^{**} satisfying $a = v|a|$ and v^*v is the range projection of $|a|$. Moreover, for each $h \in C(\sigma(|a|))$, with $h(0) = 0$ the element $vh(|a|) \in A$ (see [1, Lemma 2.1]).

In our next result we study the additional properties inherited by some particular generalized inverses.

Proposition 2.2. *Let a be an element in a C*-algebra A. The following statements hold:*

(a) *If B is a C*-subalgebra of A with $a \in B$. Then a is regular in A whenever it is regular in B;*

(*b*) *If a belongs to A_{sa} and is regular, then there exists a generalized inverse of a in A_{sa};*
(*c*) *If a is positive and regular in A, then there exists a positive normalized generalized inverse of a in A.*

Proof. (*a*) Clearly, a regular in B implies the existence of $b \in B$ such that $aba = a$ and hence a is regular in A.

(*b*) Suppose $a = a^*$ is regular in A. Then there exists $b \in A$ satisfying $aba = a$. Therefore $ab^*a = a$ and hence $a\frac{(b+b^*)}{2}a = a$.

(*c*) Suppose $a \geq 0$ is regular in A. By (*b*) be can find $b = b^*$ in A with $aba = a$. Since elements of the form x^*x are positive in A (see [58, Theorem 1.4.4]), we know that bab is a positive element in A with

$$a(bab)a = (aba)ba = aba = a$$

and

$$(bab)a(bab) = babab = bab.$$

$\square$

Let B be a C*-subalgebra of a C*-algebra A. We say that B is a *hereditary C*-subalgebra* if for $a \in A^+$ and $b \in B^+$ the inequality $a \leq b$ implies $a \in B$ (see [53, §3.2]). It is known that a B is hereditary if, and only if, $BAB \subseteq B$ (see [53, Theorem 3.2.2]). Furthermore, if $a \in A^+$ then the hereditary C*-subalgebra $A(a)$ of A generated by a coincides with the norm closure of the set aAa, that is, $A(a) = \overline{aAa}$ (see [53, Corollary 3.2.4]). It is clear that $A_a \subseteq A(a)$. Although $r(a)$ need not be in A, we know that $r(a)x = xr(a) = x$ for every $x \in A(a)$.

Let a be positive in A. For each natural number n, $a \in A_a = A_{a^n} \subseteq A(a^n)$ and the latter is a hereditary C*-subalgebra of A. Thus, $A(a) \subseteq A(a^n) \subseteq A(a)$, and we have $A(a) = A(a^n)$ and $r(a) = r(a^n)$. When $r(a)$ lies in A, then $r(a)$ belongs to $A(a)$ and is the unit element of this algebra.

We can now revisit the main connections between regularity and Moore–Penrose invertibility in a C*-algebra. The next theorem gathers some of the results originally established by L.A. Harris [32] and R. Harte and M. Mbekhta [34]. Here we present the results and we offer an alternative approach.

Theorem 2.3. ([34, Theorems 5, 6 and 7] and [32, Lemma 3.8]) *The following statements hold for every C*-algebra A.*

(*a*) *If a is positive in A, then the following assertions are equivalent*
  (*a*.1) *a is regular;*
  (*a*.2) *The range projection of a in A^{**} lies in A and a is invertible in the C*-algebra $A(r(a)) = A(a)$;*
  (*a*.3) *0 is an isolated point in $\sigma(a) \cup \{0\}$;*
  (*a*.4) *A_a is unital and a is invertible in the C*-algebra A_a;*
  (*a*.5) *a is Moore–Penrose invertible.*
(*b*) *If a is an element in A, then the following assertions are equivalent*
  (*b*.1) *a is regular;*
  (*b*.2) *0 is an isolated point in $\sigma(|a|) \cup \{0\}$;*

(b.3) *The partial isometry appearing in the polar decomposition of* a *lies in* A *and* $|a|$ *is invertible in* $A_{|a|}$*;*
(b.4) $A(|a|)$ *is unital and* $|a|$ *is invertible in* $A(|a|)$*;*
(b.5) a *is Moore–Penrose invertible.*

(c) a *is regular in* A *if, and only if,* $|a|$ *is invertible in* $A_{|a|}$*;*

(d) a *is regular in* A *if, and only if,* aa^* *(equivalently,* $|a^*|$*) is regular if, and only if,* a^*a *(equivalently,* $|a|$*) is regular;*

(e) *If* a *is regular in* A*, then* a *admits a unique Moore–Penrose inverse. The unique Moore–Penrose inverse of a regular element* a *will be denoted by* $a^\dagger$*, and in such a case* $a^\dagger = cv^*$*, where* v *is the partial isometry in the polar decomposition of* a *and* c *is the inverse of* $|a|$ *in the hereditary C*-subalgebra* $A(|a|)$*.*

Proof. (a) (a.1) $\Rightarrow$ (a.2) Suppose a is positive and regular in A. Then there exists a positive b satisfying $aba = a$ (see Proposition 2.2). Let us find $0 \leq c \in A_a$ such that $c^3 = a$. The identity $c^3bc^3 = c^3$ multiplied on the left by $(\frac{1}{n} + c)^{-1}$ implies that

$$\left((\frac{1}{n} + c)^{-1}c\right) c^2bc^3 = \left((\frac{1}{n} + c)^{-1}c\right) c^2.$$

Taking weak*-limits in n, we deduce from the separate weak*-continuity of the product of A^{**} (see [58, Theorem 1.7.8]) that

$$c^2bc^3 = r(c)c^2bc^3 = r(c)c^2 = c^2. \tag{2.1}$$

Now, multiplying both sides of the identity (2.1) on the right by $(\frac{1}{n} + c)^{-1}$ and taking weak*-limits in n we deduce that

$$c^2bc^2 = c^2bc^2\ r(c) = c\ r(c) = c. \tag{2.2}$$

We set $d = cbc$. Clearly $d \geq 0$. Applying (2.2) we get

$$cdc = c(cbc)c = c^2bc^2 = c,$$

and hence c is regular in A, and cd and dc are idempotents in A. Moreover, applying (2.2) we prove that

$$cd = c(cbc) = (c^2b)c = c^2bc^2bc^2 = (c^2bc^2)bc^2 = cbc^2 = (cbc)c = dc.$$

Therefore $(cd)^* = dc = cd = (dc)^*$, which proves that $cd = dc$ is a projection in A, and thus c is Moore–Penrose invertible. We further known that $(cd)c = c(dc) = c$, and hence $cd = dc \geq r(c) = r(a)$ as projections in A^{**}. However, since c, d are positive elements in the hereditary C*-algebra $A(a)$, we know that $r(a)cd = cd = cd\,r(a)$, which proves that $r(a) = r(c) = cd \in A$, and hence $A(a)$ is a unital C*-algebra.

Additionally, by the commutativity of c and d we also know that $d^3a = ad^3 = c^3d^3 = (cd)^3 = r(a)$, which shows that a is invertible in $A(a)$.

(a.2) $\Rightarrow$ (a.3) Let us assume that $A(a)$ is unital and a is invertible in $A(a)$ with inverse c. It is known that c lies in A_a. In this case $r(a) = ac \in A_a \cong C_0(\sigma(a))$ and the latter is a unital abelian C*-algebra where a is a positive and invertible generator. Then 0 must be isolated in $\sigma(a) \cup \{0\}$.

(a.3) $\Rightarrow$ (a.4) is clear.

For the implication $(a.4) \Rightarrow (a.5)$ we can apply again the Gelfand theory to guarantee that, 0 is isolated in $\sigma(a) \cup \{0\}$, $r(a) \in A_a$, and $A_a \cong C(\sigma(a)\backslash\{0\})$ is a unital abelian C*-algebra where a is a positive and invertible generator. Therefore, there exists a positive $b \in A_a$ with $ab = ba = r(a)$ (and then $aba = a$ with $ab = ba = r(a)$).

$(a.5) \Rightarrow (a.1)$ is clear.

(b) $(b.1) \Rightarrow (b.2)$ Suppose a is regular. Let us take b in A such that $aba = a$. Let $a = v|a|$ be the polar decomposition of a with v a partial isometry in A^{**}. The equality $aba = a$ can be written in the form $v\ |a|bv\ |a| = v\ |a|$. Multiplying on the left by v^* we get

$$|a|bv\ |a| = r(|a|)|a|bv\ |a| = v^*v\ |a|bv\ |a| = v^*v\ |a| = r(|a|)\ |a| = |a|,$$

which shows that $|a|$ is regular in A^{**}, and hence by (a), 0 is isolated in the set $\sigma_{A^{**}}(|a|) \cup \{0\} = \sigma_A(|a|) \cup \{0\}$ (see [59, Proposition I.4.8] for the last equality).

$(b.2) \Rightarrow (b.3)$ Suppose that 0 is isolated in the set $\sigma_A(|a|) \cup \{0\}$. Let $a = v|a|$ be the polar decomposition of a in A^{**}. The functions h, g defined by $h(t) = 1$ and $g(t) = \frac{1}{t}$ if $t \in \sigma(|a|)\backslash\{0\}$ and $h(0) = g(0) = 0$ if $0 \in \sigma(|a|)$, both belong to $C_0(\sigma(|a|))$, and hence $v = vh(|a|) \in A$ (see [1, Lemma 2.1]) and $c = g(|a|) \in A_{|a|}$. Actually, $h(|a|) = r(|a|)$, $a = e|a|$, $c|a| = |a|c = r(|a|)$, and $e^*e = (vh(|a|))^*(vh(|a|)) = h(|a|)v^*vh(|a|) = h(|a|)r(|a|)h(|a|) = h(a)^2 = h(a) = r(|a|)$, witnessing that $e = v \in A$.

$(b.3) \Rightarrow (b.4)$ is clear.

$(b.4) \Rightarrow (b.5)$ Suppose $a = v|a|$ is the polar decomposition of a in A^{**} with $v \in A$ and $|a|$ is invertible in $A(|a|)$ with inverse c. Since $v \in A$ we know that $v^*v = r(|a|) \in A$. We further know that $v^*v = r(|a|)$ is the unit of the C*-algebra $A(|a|)$. Taking $b = cv^*$ we arrive to

$$aba = v|a|cv^*v|a| = v|a|c|a| = v|a| = a,$$

$$bab = c(v^*v)(|a|c)v^* = cv^*vv^* = cv^* = b,$$

$ab = v|a|cv^* = vv^*vv^* = vv^*$ and $ba = cv^*v|a| = v^*v$, which proves that a is Moore–Penrose invertible.

Finally, $(b.5) \Rightarrow (b.1)$ is clear.

The assertion (c) is a clear consequence of (a) and (b), and statement (d) also follows from (a) and (b) and the fact that $\sigma(|a|)^2 \cup \{0\} = \sigma(|a|^2) \cup \{0\} = \sigma(|a^*|^2) \cup \{0\} = \sigma(|a^*|)^2 \cup \{0\}$.

(e) Suppose a is a regular element in A. We know from (b) and (a) that a is Moore–Penrose invertible, 0 is an isolated point in $\sigma(|a|) \cup \{0\}$, the partial isometry v appearing in the polar decomposition of a lies in A, $v^*v = r(|a|)$, and $|a|$ is invertible in $A(a)$. Let b be any Moore–Penrose inverse of a in A, then $aba = a$, $bab = b$, with ab, ba projections in A. Let c be the inverse of $|a|$ in $A(|a|)$. We have shown above that $d = cv^*$ is a Moore–Penrose inverse of a. Since ad, da, ba and ab are projections in A, the identities

$$ab = (ada)b = (ad)(ab) = (ab)(ad) = (aba)d = ad,$$

and

$$ba = b(ada) = (ba)(da) = (da)(ba) = d(aba) = da,$$

guarantee that $ad = ab$ and $ba = da$. Therefore

$$d = (da)d = (ba)d = b(ad) = bab = b,$$

that is $b = d = cv^*$. □

Remark 2.4. We observe that the spectrum of an element a in a C*-algebra A does not change when computed in any other C*-subalgebra B of A containing a. Therefore, by Theorem 2.3, the first statement in Proposition 2.2 is actually an equivalence.

A regular element a in a C*-algebra may admit many different generalized inverses, however they are all uniquely determined in the space v^*vAvv^*, where v is the partial isometry in the polar decomposition of a.

Proposition 2.5. *Let a be a regular element in a C*-algebra, and let b be a generalized inverse of a (i.e. $aba = a$). Suppose v is the partial isometry in the polar decomposition of a and c is the inverse of $|a|$ in $A(|a|)$. Then $(v^*v)\ b\ (vv^*) = cv^* = a^\dagger$.*

Proof. If in the equality $aba = a$ we replace a with $v|a|$, we get $v|a|bv|a| = v|a|$. Multiplying by v^* on the left we deduce that $|a|bv|a| = |a|$. Now, multiplying by c from both sides it follows that $c|a|bv|a|c = c|a|c$, and consequently

$$(v^*v)bv = (v^*v)bv(v^*v) = c|a|bv|a|c = c|a|c = c.$$

Finally, $(v^*v)b(vv^*) = (v^*vbv)v^* = cv^* = a^\dagger$. □

It is easy to deduce that for each non-zero partial isometry e in a C*-algebra A we have $e^\dagger = e^*$. We observe that, in this case, $\|e^\dagger\| = \|e^*\| = \|e\| \leq 1$. This is actually a property which characterizes partial isometries.

Theorem 2.6. [4, Theorem 2.1] *Let e be a norm-one element in a C*-algebra A. Then the following statements are equivalent:*

(a) e is a partial isometry;
(b) e is regular and $\|e^\dagger\| \leq 1$;
(c) e is regular and admits a generalized inverse b with $\|b\| \leq 1$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. We shall only prove $(c) \Rightarrow (a)$. Suppose there exists $b \in A$ with $\|b\| \leq 1$ and $ebe = e$. Applying Proposition 2.5 we derive that $(v^*v)b(vv^*) = (v^*vbv)v^* = cv^* = e^\dagger$, where $v \in A$ is the partial isometry in the polar decomposition of e and c is the inverse of $|e|$ in $A(|e|)$ (compare Theorem 2.3). By hypothesis

$$1 \geq \|b\| \geq \|(v^*v)b(vv^*)\| = \|cv^*\| = \|e^\dagger\|.$$

In particular, $1 \geq \|cv^*\|^2 = \|cv^*vc\| = \|cr(|e|)c\| = \|c^2\| = \|c\|^2$. Since c is the inverse of $|e|$ in $A_{|e|}$, with $\||e|\| = 1 \geq \|c\|$, we can easily deduce from the Gelfand representation of $A_{|e|}$ that $|e| = c$ is a projection, and thus $e = v|e|$ is a partial isometry. □

In the particular case of $A = B(H)$, the conclusion of the above theorem was previously established by M. Mbekhta in [51, Theorem 3.1].

In order to revisit other characterizations of partial isometries in terms of regularity we recall the notion of reduced minimum modulus of an operator. Let $T : X \to Y$ be a non-zero bounded (real, complex or conjugate) linear operator between two normed spaces. The *reduced minimum modulus* of T is the real number defined by

$$\gamma(T) := \inf\{\|T(x)\| : \operatorname{dist}(x, \ker(T)) \geq 1\}. \tag{2.3}$$

According to [5], for $T = 0$ we set $\gamma(T) = 0$ (in other references, like [41], $\gamma(0) = \infty$). It is known that a bounded linear operator T between Banach spaces has closed image if, and only if, $\gamma(T) > 0$ (see [41, Theorem IV.5.2]). For T invertible the reduced minimum modulus of T is $\|T^{-1}\|^{-1}$.

The reduced minimum modulus of a bounded linear operator on a complex Hilbert space is especially useful.

Lemma 2.7. ([5, page 280] and [47]) *Let H be a complex Hilbert space. For each non-zero T in $B(H)$, the reduced minimum modulus of T satisfies the following formula:*

$$\begin{aligned}\gamma(T) &= \inf\{\|T(\xi)\| : \xi \in \ker(T)^{\perp},\ \|\xi\| \geq 1\} \\ &= \inf\{\|T(\xi)\| : \xi \in \ker(T)^{\perp},\ \|\xi\| = 1\} = \inf\left(\sigma_{B(H)}(|T|)\backslash\{0\}\right).\end{aligned}$$

Proof. We shall first prove

$$\gamma(T) = \inf\{\|T(\xi)\| : \xi \in \ker(T)^{\perp},\ \|\xi\| \geq 1\}.$$

Clearly $\{\xi \in \ker(T)^{\perp},\ \|\xi\| \geq 1\} \subseteq \{\xi \in H,\ \operatorname{dist}(\xi, \ker(T)) \geq 1\}$, and hence $\gamma(T) \leq \inf\{\|T(\xi)\| : \xi \in \ker(T)^{\perp},\ \|\xi\| \geq 1\}$. To prove the reciprocal inequality, for each $\varepsilon > 0$, there exists $\xi \in H$ with $\operatorname{dist}(\xi, \ker(T)) \geq 1$ and $\gamma(T) + \varepsilon > \|T(\xi)\|$. Let p be the orthogonal projection of H onto $\ker(T)^{\perp}$. Since $(1-p)(\xi)$ satisfies $\|p(\xi)\| = \|\xi - (1-p)(\xi)\| = \operatorname{dist}(\xi, \ker(T)) \geq 1$, $p(\xi) \in \ker(T)^{\perp}$, and $(1-p)(H) = \ker(T)$, we have

$$\gamma(T) + \varepsilon > \|T(\xi)\| = \|T(p(\xi))\| \geq \inf\{\|T(\eta)\| : \eta \in \ker(T)^{\perp},\ \|\eta\| \geq 1\}.$$

The arbitrariness of ε proves the first equality. The second equality is clear.

We shall finally prove the last equality. We shall apply ideas from [47] in this part of the proof. Let $T = u|T|$ be the polar decomposition of T in $B(H)$. We observe that $\ker(|T|) \subseteq \ker(T) \subseteq \ker(T^*T) = \ker(|T|)$. Since for each ($\xi \in \ker(T)^{\perp}$ with) $\|\xi\| = 1$ we have

$$\|T(\xi)\|^2 = \langle T(\xi)|T(\xi)\rangle = \langle T^*T(\xi)|\xi\rangle = \langle |T|^2(\xi)|\xi\rangle = \||T|(\xi)\|^2 \leq \||T|^2(\xi)\|,$$

we deduce that $\gamma(T)^2 = \gamma(|T|)^2 \leq \gamma(|T|^2)$. It is known that $\overline{|T|(H)} = \ker(|T|)^{\perp}$ (just apply that $|T|$ is positive).

Let $p = r(|T|)$ denote the orthogonal projection of H onto $\ker(|T|)^{\perp} = \overline{|T|(H)}$. The mapping $|T||_{p(H)} : p(H) \to p(H)$ is injective with norm-dense range. We know that $0 < \gamma(T) = \gamma(|T|)$ if, and only if, $|T|(H) = |T|(p(H))$ is closed if, and only if, $|T||_{p(H)}$ is bijective if, and only if, $|T||_{p(H)}$ is invertible in $B(p(H))$ if, and only if, $|T|$ is regular. Thus, $\gamma(T) = 0$ if, and only if, $|T|$ is not regular if, and only if, $\inf\left(\sigma_{B(H)}(|T|)\backslash\{0\}\right)=0$. Moreover, supposing that T is regular, equivalently $|T||_{p(H)}$ is invertible in $r(|T|)B(H)r(|T|)$ with inverse R, then

$$\gamma(T)=\gamma(|T|)=\inf\{\||T|(\xi)\|:\xi\in\ker(T)^{\perp}=p(H),\ \|\xi\|=1\}$$
$$=(\text{since } R \text{ is the inverse of } |T||_{p(H)})=\frac{1}{\|R\|}=\inf\left(\sigma_{B(H)}(|T|)\backslash\{0\}\right).$$

□

Introduced by R. Harte and M. Mbekhta in [35], the *left conorm* (respectively, *right conorm*) of an element a in a Banach algebra A is the quantifier

$$\gamma^{l}(a)=\gamma(L_a)\quad(\text{respectively, } \gamma^{r}(a)=\gamma(R_a)).$$

We have already seen that a regular in a C*-algebra A implies that L_a and R_a both have closed ranges. It seems natural to ask wether the reciprocal implication is also true.

Proposition 2.8. [34, Theorem 8] *Let a be an element in a C*-algebra A. Then the following are equivalent:*

(a) *a is regular;*
(b) *L_a has closed range;*
(c) *$\gamma^{l}(a)>0$;*
(d) *R_a has closed range;*
(e) *$\gamma^{r}(a)>0$.*

Consequently, for every a in A we have $\gamma^{l}(a)=\gamma^{l}(|a|)$ and $\gamma^{r}(a)=\gamma^{r}(|a|)$.

Proof. We know that an operator T between Banach spaces has closed range if, and only if, $\gamma(T)>0$ (see [41, Theorem IV.5.2]), so $(b)\Leftrightarrow(c)$, and $(d)\Leftrightarrow(e)$ by definition. The implications $(a)\Rightarrow(b)$ and $(a)\Rightarrow(d)$ follow from Proposition 2.1. We shall only prove $(b)\Rightarrow(a)$ (the other implication is similar).

We begin with an observation $\ker(L_a)=\ker(L_{a^*a})=\ker(L_{|a|})$. Indeed if $ax=0$, then $a^*ax=0$, and since a^*a is positive we can easily check that $A_{|a|}x=A_{|a|^2}x=\{0\}$. In particular $|a|x=0$. Conversely, Let $a=v|a|$ be the polar decomposition of A. If $|a|x=0$, then $ax=v|a|x=0$ and $|a|^2x=0$. Suppose $x\in A$ with $\operatorname{dist}(x,\ker(L_a))=\operatorname{dist}(x,\ker(L_{|a|}))=\operatorname{dist}(x,\ker(L_{a^*a}))\geq 1$. By the Gelfand-Naimark axiom

$$\|L_a(x)\|^2=\|v|a|x\|^2=\|x^*|a|v^*v|a|x\|=\|x^*|a||a|x\|=\||a|x\|^2=\|L_{|a|}(x)\|^2,$$

which proves that $\gamma^{l}(a)=\gamma^{l}(|a|)$, for every $a\in A$. This proves the final statement.

$(b)\Rightarrow(a)$ If L_a has closed range (equivalently, $\gamma(L_a)>0$) then $L_{|a|}$ has closed range. Since $|a|A$ is closed and contains all polynomials expressions of the form $\alpha_2|a|^2+\alpha_3|a|^3+\ldots+\alpha_k|a|^k$, we deduce, via Stone–Weierstrass theorem, that $|a|^{\frac{1}{2}}\in A_{|a|}\subseteq|a|A$. We can therefore find b in A such that $|a|b=|a|^{\frac{1}{2}}$, and thus $|a|bb^*|a|=|a|$ witnessing that $|a|$ is regular. Theorem 2.3 assures that a is regular. □

A consequence of the previous proposition shows that an element a in a C*-algebra A is not regular if, and only if, $\gamma^l(a) = \gamma^r(a) = 0$. In other words, the left and right conorms coincide on non-regular elements of A. For regular elements we need the following theorem from [35].

Theorem 2.9. [35, Theorems 3 and 4] *Let a be a non-zero element in a C*-algebra A. Then the following statements hold:*
(a) $\gamma^l(a)^2 = \gamma^l(|a|)^2 = \gamma^l(|a|^2) = \inf\{t : t \in \sigma(a^*a)\backslash\{0\}\}$*;*
(b) $\gamma^r(a)^2 = \gamma^r(|a|)^2 = \gamma^r(|a|^2) = \inf\{t : t \in \sigma(a^*a)\backslash\{0\}\}$*;*
(c) $\gamma^l(a)^2 = \gamma^l(|a|)^2 = \gamma^l(|a|^2) = \gamma^r(a)^2 = \gamma^r(|a|)^2 = \gamma^r(|a|^2)$*;*
(d) $\gamma^l(a) = \gamma^l(a^*) = \gamma^r(a) = \gamma^r(a^*) \leq \|a\|$*;*
(e) When a is regular $\gamma^l(a) = \gamma^l(a^*) = \gamma^r(a) = \gamma^r(a^*) = \frac{1}{\|a^\dagger\|}$.

Before dealing with the proof of this theorem we note that, by the above theorem, the left and right conorms coincide on every element a in a C*-algebra A, and both quantifiers measure the distance from $\sigma(|a|)\backslash\{0\}$ to zero. The *conorm function* on A is defined by

$$\gamma : A \to \mathbb{R}_0^+,$$
$$\gamma(a) = \gamma^l(a) = \gamma^r(a).$$

This notation could cause some conflict because, by the Gelfand-Naimark theorem, A is a C*-subalgebra of some $B(H)$, where H is a complex Hilbert space, and consequently every $a \in A$ is an operator in $B(H)$ whose reduced minimum modulus is also denoted by the same symbol $\gamma(a)$. Fortunately, the previous Lemma 2.7 and Theorem 2.9 avoid any cumbersome conflict to the reader.

Proof of Theorem 2.9. (a) For a non-regular element a the equalities can be easily deduced applying Theorem 2.3 and Proposition 2.8. Let us assume that a is regular. Suppose $a = v|a|$ is the polar decomposition of a in the conditions of Theorem 2.3, with $|a|$ invertible in $A_{|a|}$ with inverse c, and $L_{|a|} : v^*vAv^*v \to v^*vAv^*v$ invertible with inverse L_c.

Proposition 2.8 implies that $\gamma^l(a) = \gamma^l(|a|)$. We have seen in the proof of the just quoted proposition that $\ker(L_a) = \ker(L_{a^*a}) = \ker(L_{|a|})$. It is not hard to see that $\ker(L_{|a|}) = \ker(L_{v^*v})$.

Let $x \in A$ with $\operatorname{dist}(x, \ker(L_{|a|})) = \operatorname{dist}(x, \ker(L_{v^*v})) \geq 1$. Since $(1 - v^*v)x \in \ker(L_{v^*v})$ and $v^*vx = x - (1 - v^*v)x$, we have $\|v^*vx\| \geq \operatorname{dist}(x, \ker(L_{v^*v})) \geq 1$, therefore

$$\gamma^l(|a|) = \inf\{\||a|x\| : \operatorname{dist}(x, \ker(L_{v^*v})) \geq 1\}$$
$$= \inf\{\||a|(v^*vx)\| : \operatorname{dist}(x, \ker(L_{v^*v})) \geq 1\} \leq \inf\{\||a|y\| : y = v^*vy,\ \|y\| \geq 1\}$$
$$\leq \inf\{\|L_{|a|}(y)\| : y = v^*vyv^*v,\ \|y\| \geq 1\} = \frac{1}{\|L_c\|} = \frac{1}{\|c\|} = \inf\{t : t \in \sigma(|a|)\backslash\{0\}\}.$$

Since $|a|$ is regular $\inf\{t : t \in \sigma(|a|)\backslash\{0\}\} > 0$. Let us take any positive λ with $\lambda < \inf\{t : t \in \sigma(|a|)\backslash\{0\}\}$. Then $|a| - \lambda v^*v$ is invertible in $A_{|a|}$ with inverse b. It is easy to check that $b|a| = |a|b = v^*v + \lambda b$ and $(v^*v + \lambda b)^*(v^*v + \lambda b) \geq \lambda^2 bb$. For each x in A, by the Gelfand-Naimark axiom we have

$$\||a|(bx)\|^2 = \|(|a|b)x\|^2 = \|(v^*v + \lambda b)x\|^2 = \|x^*(v^*v + \lambda b)^*(v^*v + \lambda b)x\|$$

$$\geq \lambda^2 \|x^* bbx\| = \lambda^2 \|bx\|^2,$$

which assures that $\left\| |a| \frac{bx}{\|bx\|} \right\| \geq \lambda$, for every $x \in A$. Since $bA = v^*vA$, we can deduce that $\lambda \leq \gamma^l(|a|)$. The arbitrariness of λ shows that $\gamma^l(|a|) \leq \inf\{t : t \in \sigma(|a|)\backslash\{0\}\}$, and thus both quantities are the same.

The rest follows from the spectral mapping theorem.

The arguments given in the proof of (a) can be easily adapted to establish (b). The affirmations (c) and (d) follow from (a) and (b). $\square$

An straight consequence of the above theorem gives another characterization of (non-zero) partial isometries.

Theorem 2.10. [35, (4.9)] *Let e be a norm-one element in a C*-algebra A. Then the following statements are equivalent:*

(a) *e is a partial isometry;*
(b) *$\gamma(e) = 1$;*
(c) *$\gamma(e) \geq 1$.*

Proof. $(a) \Rightarrow (b)$ follows from Theorem 2.9 and $e^\dagger = e^*$. $(b) \Rightarrow (c)$ is clear. Finally $(c) \Rightarrow (a)$ follows from Theorems 2.9 and 2.6. $\square$

Let us remark that Theorem 2.10 was established in [51, Theorem 3.1] in the case $A = B(H)$.

To complete the whole picture concerning the conorm on a C*-algebra, we note that R. Harte and M. Mbekhta studied in [35] the continuity of this mapping. The main conclusions are subsumed in the following result.

Theorem 2.11. [35, Theorems 7 and 9]

(a) *The conorm of every C*-algebra is upper semi-continuous on $A\backslash\{0\}$. Consequently, the conorm is continuous at non-regular elements (i.e. at elements $a \in A\backslash\{0\}$ where $\gamma(a) = 0$);*
(b) *The reduced minimum modulus is always continuous on the open sets of bounded below and of almost open operators between a pair of normed spaces X and Y. Moreover, if $T : X \to Y$ is a bounded linear operator with $\gamma(T) > 0$, $\ker(T) \neq \{0\}$, and $\overline{T(X)} \neq Y$, then T is not a continuity point of γ.* $\square$

In a recent contribution F.B. Jamjoom, H. Talawi, A.A. Siddiqui and the second author of this survey establish a new result in this direction (see [39]).

Proposition 2.12. [39, Corollary 3.7] *Let A be an extremally rich C*-algebra in the sense of [12]. Then the conorm of A is continuous at a point $a \in A$ if, and only if, either a is not regular (i.e. $\gamma(a) = 0$) or a is Brown-Pedersen quasi-invertible (i.e. a is regular and the partial isometry appearing in its polar decomposition is an extreme point of the closed unit ball of A).* $\square$

The problem of determining the continuity points of the conorm in a general C*-algebra remains open.

2.1. **Real C*-algebras.** The results concerning regularity in C*-algebras also make sense in a wider class of Banach *-algebras, the class of real C*-algebras (see [28], [48]). We recall that a *real C*-algebra* is a real Banach *-algebra satisfying the following axioms:

(1) $\|a^*a\| = \|a\|^2$, for every $a \in A$;
(2) For each $a \in A$ the element $1 + a^*a$ is invertible (in A or in its unitization if A is not unital).

A real version of the Gelfand-Naimark theorem asserts that a real Banach *-algebra A is a real C*-algebra if, and only if, it is isometrically *-isomorphic to a norm-closed real *-algebra of the real C*-algebra of all bounded linear operators on a real Hilbert space (cf. [48, Corollary 5.2.11]).

Henceforth, C*-algebras will be also called "complex C*-algebras". The class of real C*-algebras includes all complex C*-algebras.

The most favorable point of view to deal with real C*-algebras is the following construction: The algebraic complexification $A_c = A \oplus iA$, of a real C*-algebra A admits a C*-norm extending the norm of A, and there exists an involutive conjugate-linear *-automorphism τ on A_c such that

$$A = {A_c}^{\tau} := \{x \in A_c : \tau(x) = x\}$$

(see [48, Proposition 5.1.3] or [56, Lemma 4.1.13], and [28, Corollary 15.4]).

A real von Neumann algebra is a real C*-algebra whose underlying Banach space is a dual Banach space. *Real von Neumann algebras* are also called *real W*-algebras*. The bidual of a real C*-algebra is a real von Neumann algebra [17, Theorem 1.6]. It is known that every real von Neumann algebra admits a unique (isometric predual) and its product is separately weak*-continuous [38] and [50, Proposition 2.3 and Theorem 2.11].

The *complexified spectrum* (or simply the *spectrum*) of an element a in a real C*-algebra A is the spectrum of a when it is regarded as an element in A_c, that is,

$$\sigma_A(a) = \sigma_{A_c}(a)$$

(see [48, Definition 2.14] or [28, Definition in page 75]).

As in the complex case, we shall say that an element e in a real C*-algebra A is a partial isometry if $ee^*e = e$. Let A_c be the complexification of A and let τ be an involutive conjugate-linear *-automorphism on A_c such that $A = {A_c}^{\tau}$. Let $\mathcal{U}(A)$ and $\mathcal{U}(A_c)$ denote the sets of all partial isometries in A and A_c, respectively. Clearly every partial isometry in A is a partial isometry in A_c. Moreover, since τ is a *-automorphism, we can easily check that

$$\mathcal{U}(A) = \mathcal{U}(A_c)^{\tau} = \{e \in \mathcal{U}(A_c) : \tau(e) = e\}.$$

Regular elements in A are precisely the regular elements in A_c which are τ-symmetric. Projections, range projections, polar decompositions, modules, Moore–Penrose invertible elements, Moore–Penrose inverses and conorms can be literally extended to the setting of real C*-algebras (see [28], [48] and [17]).

Let a be an element in a real C*-algebra A whose complexification is denoted by A_c, and let τ be an involutive conjugate-linear *-automorphism on A_c such that $A = A_c{}^{\tau}$. It is known that $\tau^{**} : A_c^{**} \to A_c^{**}$ is an involutive conjugate-linear *-automorphism and $A^{**} = (A_c^{**})^{\tau^{**}}$. It can be easily seen that if $a = v|a|$ is the polar decomposition of a in A_c^{**}, then $v \in A^{**}$ and $|a| \in A$. Actually, $A_{|a|} = \big((A_c)_{|a|}\big)^{\tau}$. Combining these facts with the already referred Lemma 2.1 in [1], it follows that

$$vh(|a|) \in A \text{ for each } h \in C(\sigma(|a|), \mathbb{R}), \text{ with } h(0) = 0. \tag{2.4}$$

When a real C*-algebra A is regarded as the real C*-subalgebra of all τ-fixed points in its complexification for a suitable period-2 *-automorphism τ on A_c, the results in Theorems 2.3, 2.6, 2.9 and 2.10 remain valid for real C*-algebras. To save space the proofs are left to the reader. The results characterizing partial isometries in real C*-algebras are stated below.

Theorem 2.13. *Let e be a norm-one element in a real C*-algebra A. Then the following statements are equivalent:*

(a) *e is a partial isometry;*
(b) *e is regular and $\|e^\dagger\| \leq 1$;*
(c) *e is regular and admits a generalized inverse b with $\|b\| \leq 1$.* □

The real version of Theorem 2.10 reads as follows:

Theorem 2.14. *Let e be a norm-one element in a real C*-algebra A. Then the following statements are equivalent:*

(a) *e is a partial isometry;*
(b) *$\gamma(e) = 1$;*
(c) *$\gamma(e) \geq 1$.* □

3. Geometric characterization in real and complex C*-algebras

A deep geometric and natural question, motivated by the results by R.V. Kadison in [40], asks whether the partial isometries in a C*-algebra A can be recovered from the geometric Banach space structure of A alone, without recourse to the product and the adjoint operations (see [2, Introduction]). A groundbreaking result, due to C.A. Akemann and N. Weaver, asserts that a geometric characterization is always possible for C*-algebras. In this section we shall revisit their result and arguments and we shall extend the conclusion to real C*-algebras with a unified approach.

Let X be a real or complex Banach space. For each x in the unit sphere of X we consider the following sets:

$$D_1^X(x) := \Big\{ y \in X : \text{ there exists } \alpha > 0 \text{ with } \|x \pm \alpha y\| = 1 \Big\}, \tag{3.1}$$

$$D_2^X(x) := \{ y \in X : \|x + \beta y\| = \max\{1, \|\beta y\|\}, \text{ for all } \beta \in \mathbb{R} \}.$$

Clearly, for each norm-one element x in a Banach space X, the inclusion $D_2^X(x) \subseteq D_1^X(x)$ holds. We shall see later that the equality of these two sets will have important consequences in the case in which X is a real or complex C*-algebra.

The following lemma seems to be explicitly uncovered by the available literature for C*-algebras. It should be noticed that a generalized version for JB*-triples appears in [26, Lemma 1.6].

Lemma 3.1. *Let e be a partial isometry in a real C*-algebra A. Suppose x is a norm-one element in A with $ee^*xe^*e = e$. Then $x = e + (1 - ee^*)x(1 - e^*e)$.*

Proof. Since $x = ee^*xe^*e + ee^*x(1 - e^*e) + (1 - ee^*)xe^*e + (1 - ee^*)x(1 - e^*e)$, the desired statement will follow as soon as we show that $ee^*x(1 - e^*e) = 0 = (1 - ee^*)xe^*e$.

From $\|x\| = 1$ we have that $\|ee^*x\|^2 \leq 1$. We recall that in a real or complex unital C*-algebra B, the equality $1 + \|a\| = \|1 + a\|$ holds for every positive $a \in B$. Having in mind that ee^*Aee^* is a real C*-algebra with identity ee^*, a double application of the Gelfand-Naimark axiom gives

$$1 + \|ee^*x(1 - e^*e)\|^2 = 1 + \|ee^*x(1 - e^*e)(1 - e^*e)x^*ee^*\|$$

$$= \|ee^* + ee^*x(1 - e^*e)(1 - e^*e)x^*ee^*\| = \|e + ee^*x(1 - e^*e)\|^2$$

$$= \|ee^*xe^*e + ee^*x(1 - e^*e)\|^2 = \|ee^*x\|^2 \leq 1,$$

witnessing that $ee^*x(1 - e^*e) = 0$. The identity $(1 - ee^*)xe^*e = 0$ can be analogously obtained. $\square$

Suppose e is a partial isometry in a real C*-algebra A. Then e is an extreme point of the closed unit ball of the space ee^*Ae^*e. Indeed, having in mind that ee^*Aee^* is a real C*-algebra with identity ee^*, given $z = ee^*ze^*e \in ee^*Ae^*e$ such that $\|e \pm z\| \leq 1$, we have $zz^* \in ee^*Aee^*$ and thus

$$1 + \|z\|^2 = \|ee^* + zz^*\| = \left\| \frac{1}{2}(ee^* + ze^* + ez^* + zz^*) + \frac{1}{2}(ee^* - ze^* - ez^* + zz^*) \right\|$$

$$\leq \frac{1}{2}(\|e + z\|^2 + \|e - z\|^2) \leq 1,$$

which forces z to be zero.

We recall that elements a, b in a real or complex C*-algebra A are said to be *orthogonal* (written $a \perp b$) if $ab^* = b^*a = 0$. Let $a \perp b$ in A. By the (real or complex) Gelfand-Naimark theorem, A can be regarded as a norm closed self-adjoint subalgebra of some $B(H)$, where H is a real Hilbert space. We can therefore assume that a, b are operators in $B(H)$ with $ab^* = b^*a = 0$. For each $\xi, \eta \in H$, we have $\langle a(\xi) | b(\eta) \rangle = \langle b^*a(\xi) | \eta \rangle = 0$, and similarly $\langle a^*(\xi) | b^*(\eta) \rangle = 0$. This shows that $\overline{a(H)} \perp \overline{b(H)}$, $\overline{a^*(H)} \perp \overline{b^*(H)}$ in the Hilbert sense. It can be easily checked that

$$\|a + b\| = \max\{\|a\|, \|b\|\}. \tag{3.2}$$

We are now in position to present a geometric characterization of partial isometries in real C*-algebras. This characterization was obtained by C. Akemann and N. Weaver in the setting of complex C*-algebras (see [2, Theorem 1]). The proof presented here owes so much from their original arguments.

Theorem 3.2. [2, Theorem 1] *Let e be a norm-one element in a real or complex C^*-algebra A. Then e is a partial isometry if, and only if, $D_1^A(e) = D_2^A(e)$. Moreover, for each partial isometry e in A we have $D_1^A(e) = (1-ee^*)A(1-e^*e)$.*

Proof. ($\Rightarrow$) Let e be a partial isometry. Clearly $D_2^A(e) \subseteq D_1^A(e)$, so we only have to show the reverse inclusion.

Given $y \in D_1^A(e)$ (i.e. $\|e \pm \alpha y\| = 1$ for some $\alpha > 0$, and hence $\|\alpha y\| = 1$) we have that $\|e \pm \alpha ee^*ye^*e\| \leq 1$ and the extremality of e in ee^*Ae^*e forces to $ee^*ye^*e = 0$. Since $ee^*(e+\alpha y)e^*e = e$, Lemma 3.1 assures that

$$e + \alpha y = e + (1-ee^*)(e+\alpha y)(1-e^*e) = e + \alpha(1-ee^*)y(1-e^*e),$$

and thus $e \perp y = (1-ee^*)y(1-e^*e)$. Therefore y belongs to $D_2^A(e)$ (cf. (3.2)).

($\Leftarrow$) Suppose now that e is not a partial isometry. Let $e = u|e|$ be the polar decomposition of e. Clearly $|e|$ is not a projection, otherwise e coincides with the partial isometry u. Let $t_0 \in]0,1[\cap\sigma(|e|)$ and $h : \sigma(|e|) \to \mathbb{R}$ defined by $h(0) = 0$, $h(1) = 0$, $h(t_0) = 1 - t_0$ and affine elsewhere. Let us set $y = uh(|e|)$. Clearly, $y \in A$ (see (2.4)). It is straightforward to check, via continuous functional calculus at the element $|e|$, that $\|e \pm y\| = \||e| \pm h(|e|)\| = 1$ (i.e. y belongs to $D_1^A(e)$). However, y does not belong to $D_2^A(e)$. Namely, if we take, for example, $\beta = \frac{1}{1-t_0}$, we have $\|e + \beta y\| = \||e| + \beta h(|e|)\| = 1 + t_0 > 1$, $\|e\| = 1$ and $\|\beta y\| = 1$.

The final statement have been obtained in the first part of the proof. $\square$

The *orthogonal set* of an element x in a real C*-algebra A is defined as

$$\{x\}^\perp := \{y \in A : x \perp y\}.$$

Clearly, $\{x\}^\perp \subseteq D_2^A(e)$ (see (3.2)). We can now obtain a characterization of partial isometries in terms of the orthogonal complement.

Theorem 3.3. *Let e be a norm-one element in a real or complex C^*-algebra A. Then e is a partial isometry if, and only if, $\{e\}^\perp = (1-ee^*)A(1-e^*e)$.*

Proof. ($\Rightarrow$) Suppose e is a partial isometry and take $b \in \{e\}^\perp$. Since $be^* = 0$ and hence $be^*e = 0$, we have $b = b(1-e^*e)$ and similarly $b = (1-ee^*)b$. We therefore have $b = (1-ee^*)b(1-e^*e) \in (1-ee^*)A(1-e^*e)$.

($\Leftarrow$) If e is not a partial isometry, then e^*e is a positive element which is not a projection. The element $x = (1-ee^*)e(1-e^*e) \in (1-ee^*)A(1-e^*e)$, and $x^*e = (1-e^*e)e^*(1-ee^*)e = |e|^2(1-|e|^2)^2 \neq 0$ because $e^*e = |e|^2$ is not a projection. $\square$

We can now recover the classical characterization of the extreme points of the closed unit ball of a C*-algebra due to R.V. Kadison (see [40, Theorem 1]).

We recall that a partial isometry e in a real C*-algebra A is called *complete* if $(1-ee^*)A(1-e^*e) = \{0\}$.

Corollary 3.4. ([40, Theorem 1], [58, Theorem 1.6.4]) *The set of extreme points of the closed unit ball of a real or complex C^*-algebra, A, are precisely the complete partial isometries in A. More concretely, the following statements are equivalent for every norm-one element e in a real or complex C^*-algebra A.*

(*a*) *e is an extreme point of the closed unit ball of A;*
(*b*) $D_1^A(e) = \{0\}$*;*
(*c*) *e is a partial isometry and* $(1 - ee^*)A(1 - e^*e) = \{0\}$*;*
(*d*) $(1 - ee^*)A(1 - e^*e) = \{0\}$.

Proof. $(a) \Leftrightarrow (b)$ If e is an extreme point of the closed unit ball of A, then trivially $D_1^A(e)$ must be $\{0\}$. Conversely, if $D_1^A(e) = \{0\}$, and $e = \frac{1}{2}(x + y)$ with $\|x\|, \|y\| = 1$, then taking $z = \frac{1}{2}(x - y)$ we have $\|z\| \leq 1$, $\|e + z\| = \|x\| = 1$ and $\|e - z\| = \|y\| = 1$. This implies that $z \in D_1^A(e) = \{0\}$, and hence $x = y = e$.

$(b) \Rightarrow (c)$ If $D_1^A(e) = \{0\}$, then $D_2^A(e) = D_1^A(e) = \{0\}$. Theorem 3.2 gives the desired statement.

The implication $(c) \Rightarrow (d)$ is clear. Finally, for $(d) \Rightarrow (b)$, we observe that $\{e\}^\perp \subseteq (1 - ee^*)A(1 - e^*e) = \{0\}$, and hence Theorems 3.2 and 3.3 prove (b). □

In [9, 10] D.P. Blecher and M. Neal established a metric characterization of unitaries, isometries, and coisometries in terms of the operator space structure of C*-algebras and TRO's. We recall that a *ternary ring of operators* (a *TRO* in the terminology of D.P. Blecher and M. Neal in [8] and M. Neal and B. Russo in [54]) is a closed subspace Z of a C*-algebra A such that $ab^*c \in Z$, for every $a, b, c \in Z$.

For each complex Hilbert space H, and each natural number n, the symbol H^n stands for the direct sum of n copies of H. According to this notation, the space $M_n(B(H))$ of all $n \times n$-matrices with entries in $B(H)$ can be naturally identified with $B(H^n)$. The operator norm $\|.\|_n$ is the norm of $M_n(B(H))$ when the latter space is identified with $B(H^n)$. Given an operator space $Z \subset B(H)$, the space $M_n(Z)$ can be regarded as a subspace of $M_n(B(H))$, and consequently the operator norm $\|.\|_n$ defines a norm on $M_n(Z)$. The operator norm, $\|.\|_{m,n}$, on spaces $M_{mn}(Z)$, of nonsquare matrices with entries in Z, is obtained from the operator norm by completing with columns or rows of zeroes to get an square matrix.

Let $Z \subset B(H)$ and $X \subset B(H')$ be operator spaces. A bounded linear operator $T : Z \to X$ admits an extension to a bounded linear map $T_n : M_n(Z) \to M_n(X)$, given by $T_n((x_{ij})) = (T(x_{ij}))$. The operator T is *completely bounded* (respectively, *completely contractive*, *completely isometric*) if $\sup_n \|T_n\| < \infty$ (respectively, $\|T_n\| \leq 1$ for all n, T_n is an isometry for all n).

An element u is a TRO Z is a called a *coisometry* (respectively, an *isometry*) if $uu^*z = z$ (respectively, $zu^*u = z$) for all $z \in Z$. We say that u is a *unitary* if u is an isometry and a coisometry. If Z is a C*-algebra, these definitions coincide with the usual notions of unitaries, coisometries, and isometries.

Accordingly to [9], an element v in an operator space X is called a *unitary* (respectively, an *isometry*, or a *coisometry*) in X if there exists a complete isometry T from X into some $B(H)$ such that $T(v)$ a unitary (respectively, an isometry or a coisometry). It is known that when C*-algebras and TROs are regarded as operator spaces these definitions are perfectly compatible with those given in the previous paragraph (see [9, Lemma 2.3]).

Given an element u in an operator space X, the symbol $u_n \in M_n(X)$ will stand for the diagonal matrix with value u at every diagonal entry.

After recalling the basic notation on operator spaces, we can revisit the metric characterization obtained by D.P. Blecher and M. Neal.

Theorem 3.5. [9, Theorem 2.4] *An element u in an operator space X is a unitary in X if, and only if, $\left\|\begin{pmatrix} u_n & x \end{pmatrix}\right\|_{n,2n}^2 = 1+\|x\|_n^2$ and $\left\|\begin{pmatrix} u_n & x \end{pmatrix}^t\right\|_{2n,n}^2 = 1+\|x\|_n^2$, for all $x \in M_n(X)$ and $n \in \mathbb{N}$. Indeed, it suffices to consider norm one matrices x here. Similarly, u is a coisometry (respectively, isometry) in X if, and only if, the first (respectively, the second) of these norm conditions holds for all $x \in M_n(X)$.* $\square$

4. A Jordan approach: von Neumann regularity in JB*-triples, triple spectrum, Quadratic conorm

One of the most successful generalizations of C*-algebras appears in the study of the theory of holomorphic functions on arbitrary complex Banach spaces and the extension of the Riemann mapping theorem to arbitrary dimensions. The studies to classify bounded symmetric domains in arbitrary complex Banach spaces led W. Kaup to introduce the notion of JB*-triple in [43].

A *complex* (respectively, *real*) *Jordan triple system* is a complex (respectively, real) linear space E where there exists a triple product

$$\{.,.,.\} : E \times E \times E \to E$$
$$(x,y,z) \mapsto \{x,y,z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one (respectively, trilinear) and satisfies the so-called *Jordan identity*

$$L(x,y)\{a,b,c\} = \{L(x,y)a,b,c\} - \{a,L(y,x)b,c\} + \{a,b,L(x,y)c\}, \tag{4.1}$$

for all $x,y,a,b,c \in E$, where $L(x,y) : E \to E$ is the linear mapping given by $L(x,y)z = \{x,y,z\}$. For every couple of element a,b in a complex (respectively, real) Jordan triple system E, $Q(a,b)$ will stand for the conjugate (respectively, real) linear operator defined by $Q(a,b)(x) := \{a,x,b\}$. By the above axioms we have $Q(a,b) = Q(b,a)$ for every $a,b \in E$. The mapping $Q(a,a)$ will be simply denoted by $Q(a)$. It follows from [**?**] that the identities

$$Q(a)Q(b)Q(a) = Q(Q(a)b), \tag{4.2}$$

and

$$Q(a,Q(a)(b)) = Q(a)L(b,a) = L(a,b)Q(a), \tag{4.3}$$

hold for every $a,b \in E$ (see [49, Appendix A1]).

Accordingly to the definition introduced by W. Kaup in [43], a (complex) *JB*-triple* is a complex Jordan triple system which is also a Banach space satisfying the following axioms:

(1) For each x in E the map $L(x,x)$ is an hermitian operator with non-negative spectrum;
(2) $\|\{x,x,x\}\| = \|x\|^3$, for all $x \in E$.

Every C*-algebra is a JB*-triple with respect to the product

$$\{x,y,z\} := \frac{1}{2}(xy^*z + zy^*x), \tag{4.4}$$

and the Banach space $B(H,K)$ of all bounded linear operators between two complex Hilbert spaces H,K is a JB*-triple with the product given in (4.4). In the setting of Jordan algebras, every JB*-algebra is a JB*-triple with triple product

$$\{a,b,c\} := (a\circ b^*)\circ c + (c\circ b^*)\circ a - (a\circ c)\circ b^*.$$

By a JC*-triple we mean a JB*-subtriple of $B(H)$. In [31, 32] L. Harris employs the term J*-algebra instead of JC*-triple. The class of TRO's considered in the previous section is strictly contained in the class of JB*-triples.

One of the consequences of the Gelfand-Naimark representation theorem for JB*-triples established by Y. Friedman and B. Russo in [27] implies that the triple product of a JB*-triple E is always contractive, that is,

$$\|\{x,y,z\}\| \leq \|x\|\,\|y\|\,\|z\|, \tag{4.5}$$

for all $x,y,z\in E$ (see [27, Corollary 3]).

Surjective isometries between JB*-triples are triple isomorphisms, and reciprocally, triple isomorphisms are surjective isometries (see [43, Proposition 5.5]). Actually, it is relatively easy to see that a continuous linear triple homomorphism Φ between JB*-triples E and F is contractive. Namely, take x in E, then

$$\|\Phi(x)\|^3 = \|\{\Phi(x),\Phi(x),\Phi(x)\}\| = \|\Phi(\{x,x,x\})\| \leq \|\Phi\|\|x\|^3,$$

which proves that $\|\Phi\|^3 \leq \|\Phi\|$.

Let us observe that from (4.5) $\|Q(a)\| \leq \|a\|^2$. Since for a non-zero a, $\|a\|^3 = \|Q(a)(a)\| \leq \|Q(a)\|\,\|a\|$, we have that $\|Q(a)\| = \|a\|^2$.

An element e in a JB*-triple E is called a *tripotent* if $\{e,e,e\} = e$. If a C*-algebra is regarded as a JB*-triple with the product defined in (4.4), an element e in A is a tripotent if, and only if, it is a partial isometry.

Suppose e is a tripotent in a JB*-triple E. Then the operator $L(e,e): E\to E$ has eigenvalues $\{0,\frac{1}{2},1\}$ and the corresponding eigenspaces induce a *Peirce decomposition* of E in the form

$$E = E_2(e)\oplus E_1(e)\oplus E_0(e),$$

where for $j=0,1,2$, $E_j(e) = \{x\in E : L(e,e)(x) = \frac{j}{2}x\}$. Given $j\in\{0,1,2\}$, the symbol $P_j(e)$ will stand for the natural projection of E onto $E_j(e)$, which is called the Peirce j-projection associated with e. It is known that

$$P_2(e) = Q(e)^2,\ \ P_1(e) = 2(L(e,e) - Q(e)^2),\ \text{ and } P_0(e) = Id_E - 2L(e,e) + Q(e)^2,$$

and $\|P_j(e)\|\leq 1$, for all $j\in\{0,1,2\}$ (see [26, Corollary 1.2]).

The Peirce arithmetic assures that the following multiplication rules hold:

$$\{E_i(e),E_j(e),E_k(e)\}\subseteq E_{i-j+k}(e),$$

if $i-j+k\in\{0,1,2\}$ and is zero otherwise, and

$$\{E_2(e),E_0(e),E\} = \{E_0(e),E_2(e),E\} = 0.$$

It is also known that the Peirce-2 subspace $E_2(e)$ is a unital JB*-algebra with unit e, product $a \circ_e b = \{a, e, b\}$ and involution $a^{\sharp_e} = \{e, a, e\}$ (cf. [11, Theorem 2.2] and [45, Theorem 3.7]).

The Jordan version of the geometric property presented in (3.2) assures that

$$\|x_0 + x_2\| = \max\{\|x_0\|, \|x_2\|\}, \tag{4.6}$$

for all $x_0 \in E_0(e)$ and $x_2 \in E_2(e)$ (cf. [26, Lemma 1.3]).

In order to have a substitute for the polar decomposition in a C*-algebra, the JB*-triple theory offers the continuous functional calculus as an alternative. More precisely, the odd powers of an element a in a JB*-triple E are defined as follows: $a^{[1]} := a$, $a^{[3]} := \{a, a, a\}$, and $a^{[2n+1]} := \left\{a, a^{[2n-1]}, a\right\} = \{a, a, a^{[2n-1]}\}$ ($n \in \mathbb{N}$). Let us recall that by the Jordan identity, Jordan triple systems are power associative, that is, $\left\{a^{[k]}, a^{[l]}, a^{[m]}\right\} = a^{[k+l+m]}$ ($k, l, m \in 2\mathbb{N} + 1$). The JB*-subtriple generated by the element a (i.e. the norm closure in E of all odd polynomials in a) is denoted by the symbol E_a.

The local Gelfand theory for JB*-triples assures the existence of a unique locally compact Hausdorff space Λ_a contained in $[0, \|a\|]$ satisfying that $\Lambda_a \cup \{0\}$ is compact and there exists a JB*-triple isomorphism (and hence an isometry) $\Phi : E_a \to C_0(\Lambda_a)$ such that $\Phi(a)(t) = t$ for all $t \in \Lambda_a$, where $C_0(\Lambda_a)$ denotes the commutative C*-algebra of all complex-valued continuous functions on $\Lambda_a \cup \{0\}$ vanishing at 0 (cf. [42, Corollary 4.8], [43, Corollary 1.15] see also [61] and [16]). The set $\mathrm{Sp}(a) = \Lambda_a \cup \{0\}$ will be called the *triple spectrum* of a. Accordingly to the above definition $0, \|a\| \in \mathrm{Sp}(a)$ and $\mathrm{Sp}(0) = \{0\}$. We shall also write $C_0(\mathrm{Sp}(a))$ instead of $C_0(\Lambda_a)$. Let us note that if 0 is isolated in the triple spectrum of a then $E_a \cong C_0(\mathrm{Sp}(a)) \cong C(\mathrm{Sp}(a)\backslash\{0\})$ the unital and commutative C*-algebra of all complex-valued continuous functions on the compact set $\mathrm{Sp}(a)\backslash\{0\} = \Lambda_a$.

The triple spectrum of an element a does not change when computed with respect to any JB*-triple containing the element a. For each odd polynomial $p(\zeta) = \alpha_1\zeta + \alpha_3\zeta^3 + \ldots + \alpha_{2n+1}\zeta^{(2n+1)}$ the symbol $p_t(a)$ will denote the element $\alpha_1 a + \alpha_3 a^{[3]} + \ldots + \alpha_{2n+1} a^{[2n+1]}$. Consequently, for each odd polynomial $p(\zeta)$, since $p_t(a)$ lies in E_a, and we conclude that the identity $\mathrm{Sp}(p_t(a)) = p(\mathrm{Sp}(a))$ holds for every $a \in E$. Actually if $f \in C_0(\mathrm{Sp}(a))$, the element $f_t(a) = \Phi^{-1}(f) \in E_a$ and

$$\mathrm{Sp}(f_t(a)) = f(\mathrm{Sp}(a)). \tag{4.7}$$

The previous identity can be considered as a triple spectral mapping. We use the notation $f_t(a)$ to avoid confusions with the usual functional calculus. A germinal triple continuous functional calculus was established by L.A. Harris in [31, Proposition 1] and [32, §3].

A *JBW*-triple* is a JB*-triple which is also a dual Banach space. Every JBW*-triple admits a unique (isometric) predual and its triple product is separately weak* continuous (see [7]). It is also known that the bidual space E^{**} of a JB*-triple is a JBW*-triple with a triple product and norm which are extensions of the triple product and norm of E, respectively (cf. [19]). The set of tripotents in a general JB*-triple may be empty. However, since the extreme points of the closed unit ball of a JB*-triple are all tripotents (see [11, Lemma 4.1] or Corollary 5.4

below), the Krein–Milman theorem guarantees that every JBW*-triple contains an abundant set of tripotents. There is a perfect analogy between the categories "C*-algebras — von Neumann algebras" and "JB*-triples — JBW*-triples".

One of the consequences of the above Gelfand theory implies that every element a in a JB*-triple E admits a cubic root (or more generally an odd root), that is there exists $b \in E_a$ such that $\{b, b, b\} = a$ (more generally, for each natural number n, there exists (a unique) $a^{[1/(2n-1)]}$ in E_a satisfying $(a^{[1/(2n-1)]})^{[2n-1]} = a$). In these conditions, the sequence $(a^{[1/(2n-1)]})$ converges in the weak*-topology of E^{**} to a tripotent, denoted by $r(a)$, which is called the *range tripotent* of a. Actually, $r(a)$ is the smallest tripotent e in E^{**} satisfying that a is positive in the JBW*-algebra $E_2^{**}(e)$. When $\|x\| = 1$, by replacing odd roots with odd powers we get another sequence, $(x^{[2n-1]})$, converging in the weak*-topology of E^{**} to a tripotent (called the *support tripotent* of x) $u(x)$ in E^{**}, which satisfies $u(x) \leq x \leq r(x)$ in the natural order of the JBW*-algebra $E_2^{**}(r(x))$ (compare [21, Lemma 3.3], see also [20, Lemma 3.2]).

For a non-zero element a in a JB*-triple E, the operator $L(a, a)$ maps E_a into itself, and hence $L(a, a)|_{E_a}$ is an element in the complex Banach algebra $B(E_a)$ whose spectrum is precisely

$$\sigma_{B(E_a)}(L(a, a)|_{E_a}) = \{t \in \mathbb{R} : (L(a, a) - t\mathrm{Id}_E)|_{E_a} \text{ is not invertible in } B(E_a)\}\,.$$

Another laborious application of the Gelfand theory assures that

$$\mathrm{Sp}(a) = \{t \in \mathbb{R}_0^+ : t^2 \in \sigma_{B(E_a)}(L(a, a)|_{E_a})\} \cup \{0\} \tag{4.8}$$

(cf. [44, Corollary 3.4], see also [16, §3]).

Let a, b be two elements in a real or complex JB*-triple E. Following standard notation, we shall say that a and b are *orthogonal* (and we write $a \perp b$) if $L(a, b) = 0$ (or equivalently $L(b, a) = 0$) holds. By the rules contained in Peirce arithmetic, $a \perp b$ whenever $a \in E_2(e)$ and $b \in E_0(e)$ for a tripotent $e \in E$. We refer to [14, Lemma 1] for different reformulations of the property "being orthogonal". Among the different reformulations in the just quoted reference, we highlight that $a \perp b$ in E if, and only if, $r(a) \perp r(b)$ in E^{**}, and in such a case $b \in E_0^{**}(r(b))$ and $a \in E_0^{**}(r(b))$, which implies that

$$\|a \pm b\| = \max\{\|a\|, \|b\|\} \quad \text{(see (4.6))}. \tag{4.9}$$

Back to the notion of regular elements, a generalization of this concept in the setting of Jordan triple systems was conducted by M. Koecher [46] and A. Fernández López, E. García Rus, E. Sánchez Campos, and M. Siles Molina [23]. We recall that an element a in a JB*-triple E is (*von Neumann*) *regular* if a belongs to $Q(a)(E)$ (i.e. there exists b in E such that $Q(a)(b) = a$) and *strongly* (*von Neumann*) *regular* if $a \in Q(a)^2(E)$.

We begin our study on regularity in JB*-triples with a generalization of Proposition 2.1 to the triple setting.

Proposition 4.1. [15] *Let a be a regular element in a JB*-triple E. Then the operator $Q(a)$ has closed range.*

Proof. Let b be an element in E such that $Q(a)(b) = a$. Applying (4.2) we get

$$Q(a)Q(b)Q(a) = Q(Q(a)(b)) = Q(a).$$

Therefore, $Q(a)Q(b)$ and $Q(b)Q(a)$ are idempotents elements in the Banach algebra $B(E)$ with $(Q(a)Q(b))Q(a) = Q(a) = Q(a)(Q(b)Q(a))$.

Let us assume that $Q(a)(x_n) \to z$ in norm for a suitable sequence (x_n) in E. Clearly, $Q(b)Q(a)(x_n) \to Q(b)(z)$ in norm. Since $Q(a)Q(b)Q(a) = Q(a)$, we obtain $Q(a)(x_n) = Q(a)Q(b)Q(a)(x_n) \to Q(a)Q(b)(z)$, which assures that $Q(a)Q(b)(z) = z \in Q(a)(E)$, as desired. □

Suppose F is a JB*-subtriple of a JB*-triple E. Let a be an element in F. Clearly, a regular in F implies that a is regular in E. The other implication is a non-trivial question.

For completeness reasons we explore next when an element a in a JB*-triple is regular in the JB*-subtriple E_a.

Lemma 4.2. *Let a be an element in a JB*-triple E. Then a is regular in E_a if, and only if, 0 is an isolated point in $Sp(a)$ if, and only if, a is invertible in the commutative C*-algebra $E_a \cong C_0(Sp(a)) \cong C(Sp(a)\backslash\{0\})$.*

Proof. We shall identify E_a with $C_0(\mathrm{Sp}(a))$ and a with the function $t \mapsto t$ for all $t \in \mathrm{Sp}(a)$. We shall only prove the first equivalence, the second one can be derived with the same arguments.

($\Leftarrow$) If 0 is an isolated point in $\mathrm{Sp}(a)$. The function $b : t \mapsto \frac{1}{t}$ $(t \in \mathrm{Sp}(a))$ is an element in E_a and $Q(a)(b) = a$.

($\Rightarrow$) Suppose a is regular in E_a. Then there exists a continuous function $b \in C_0(\mathrm{Sp}(a))$ such that $Q(a)(b) = a$. Therefore, $b(t)t^2 = t$ for all $t \in \mathrm{Sp}(a)$. The condition $b \in C_0(\mathrm{Sp}(a))$ implies that 0 must be isolated in $\mathrm{Sp}(a)$. □

Regular elements are intrinsically related to inner ideals. We recall that a closed subspace I of a JB*-triple E is called an *inner ideal* of E if $\{I, E, I\} \subseteq I$. For each element a in E, $E(a)$ will denote the norm-closure of $\{a, E, a\} = Q(a)(E)$ in E. It is known that $E(a)$ coincides with the norm-closed inner ideal of E generated by a (see [13, Proposition 2.1 and comments prior to it]). Obviously $E_a \subseteq E(a)$. The inner ideal $E(a)$ has additional properties, namely, L.J. Bunce, Ch.-H. Chu and B. Zalar show in [13, Proposition 2.1] that $E(a)$ is a JB*-subalgebra of the JBW*-algebra $E(a)^{**} = \overline{E(a)}^{w^*} = E_2^{**}(r(a))$ and contains a as a positive element.

In order to provide a JB*-triple analogue of Theorem 2.3 we gather, in the next theorem, some results borrowed from [23] and [44].

Theorem 4.3. ([32], [23] and [44]) *Let a be an element in a JB*-triple E. The following are equivalent:*

a) a is regular;
b) The cubic root of a in E_a is strongly regular in E;
c) 0 is isolated in the triple spectrum of the cubic root of a in E_a;
d) 0 is isolated in $Sp(a)$;
e) 0 is isolated in $\sigma_{B(E_a)}(L(a,a)|_{E_a}) \cup \{0\}$;

f) *The range tripotent $r(a)$ lies in E and a is an invertible element in the commutative C^*-algebra E_a;*
g) *The range tripotent $r(a)$ lies in E and there exists an element b in $E_2(r(a))$ satisfying the following properties $Q(a)(b) = a$, $Q(b)(a) = b$ and $L(b,a)P_2(r(a)) = L(a,b)P_2(r(a)) = P_2(r(a))$;*
h) *There exists an element b in E satisfying the following properties $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)\,Q(b) - Q(b)\,Q(a) = 0$;*
i) *a is strongly regular;*

Proof. We can assume that a and all the other elements in the statement are non-zero, otherwise the equivalences are clear.

$(a) \Rightarrow (b)$ Suppose a is regular in E, then there exists $b \in E$ such that $a = Q(a)(b)$. Take $c \in E_a$ such that $c^{[3]} = a$. Set $z = c - Q(c)^2(b) = c - Q(c)Q(c)(b)$. We observe that $Q(c)(z) = Q(c)(c - Q(c)^2(b)) = Q(c)(c) - Q(c)^3(b) =$ (by (4.2)) $= c^{[3]} - Q(c^{[3]})(b) = a - Q(a)(b) = 0$. Now, applying (4.2), (4.3) and the above equality, we deduce that

$$\{z,z,z\} = Q(z)(z) = Q(c - Q(c)^2(b))(z) = Q(c)(z) + Q(Q(c)^2(b))(z)$$

$$-2Q(c, Q(c)Q(c)(b))(z) = 0 + Q(c)Q(Q(c)(b))Q(c)(z) + L(c, Q(c)(b))Q(c)(z) = 0,$$

which shows that $0 = z = c - Q(c)^2(b)$, witnessing that c is strongly regular.

$(b) \Rightarrow (c)$ Let us take $c \in E_a$ such that $c^{[3]} = a$ and c is strongly regular in E. We can therefore find $b \in E$ such that $Q(c)^2(b) = c$. We set $d = Q(c)(b)$. It is easy to check that $Q(c)(d) = Q(c)^2(b) = c$, and applying (4.2) we get

$$\begin{aligned} Q(c)(d^{[3]}) &= Q(c)Q(d)d = Q(c)Q(Q(c)(b))Q(c)b \\ &= Q(Q(c)^2(b))(b) = Q(c)(b) = d. \end{aligned}$$

Under these circumstances

$$Q(d) = Q(Q(c)(d^{[3]})) = Q(c)Q(d^{[3]})Q(c) = Q(c)Q(d)^3Q(c),$$

which gives

$$\begin{aligned} Q(c)Q(d)^2 &= Q(c)Q(c)Q(d)^3Q(c)Q(c)Q(d)^3Q(c) = Q(c)^2Q(d^{[3]})Q(c)^2Q(d^{[3]})Q(c) \\ &= Q(c)Q(Q(c)(d^{[3]}))Q(c)Q(d^{[3]})Q(c) = Q(c)Q(d)Q(c)Q(d^{[3]})Q(c) \\ &= Q(Q(c)(d))Q(d^{[3]})Q(c) = Q(c)Q(d^{[3]})Q(c) = Q(Q(c)(d^{[3]})) = Q(d). \end{aligned}$$

Combining the above identities, we have

$$\begin{aligned} Q(c)Q(d) &= Q(c)Q(c)Q(d)^3Q(c) = Q(c)(Q(c)Q(d)^2)Q(d)Q(c) \\ &= Q(c)Q(d)^2Q(c) = Q(d)Q(c). \end{aligned} \tag{4.10}$$

We observe that $Q(c)Q(d)Q(c)(x) = Q(Q(c)(d))(x) = Q(c)(x)$, for every $x \in E$. This proves that

$$(Q(d)Q(c))Q(c)(x) = (\text{by } (4.10)) = (Q(c)Q(d))Q(c)(x) = Q(c)(x),$$

for every x in E. Since $E(c)$ is the norm closure of $Q(c)(E)$, we deduce, from the continuity of $Q(c)$ and $Q(d)$, that

$$Q(c)Q(d)(z) = Q(d)Q(c)(z) = z, \tag{4.11}$$

for all $z \in E(c)$. Clearly, $Q(c)(E(c)) \subseteq E(c)$. The identity in (4.11) implies that

$$\|z\| \leq \|Q(d)\| \|Q(c)(z)\|, \tag{4.12}$$

for all $z \in E(c)$. If 0 were non-isolated in $\mathrm{Sp}(c)$, we could consider the following sequence $(x_n) \subset E_c \cong C_0(\mathrm{Sp}(c)) \subseteq E(c)$ defined by

$$x_n(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2n}] \cap (\mathrm{Sp}(c)), \\ \text{affine}, & \text{if } t \in [\frac{1}{2n}, \frac{1}{n}] \cap \mathrm{Sp}(c), \\ \frac{1}{t}, & \text{if } t \in [\frac{1}{n}, \infty) \cap \mathrm{Sp}(c). \end{cases}$$

We note that $\|Q(c)(x_n)\| = 1$. Then, it would follow from (4.12) that

$$n = \|x_n\| \leq \|Q(d)\| \|Q(c)(x_n)\| = \|Q(d)\|,$$

for every natural number n, which is impossible.

The implication $(c) \Rightarrow (d)$ follows from the triple spectral mapping theorem (see (4.7)), which in particular asserts that $\mathrm{Sp}(a) = \mathrm{Sp}(c^{[3]}) = \mathrm{Sp}(a)^3$.

The equivalence $(d) \Leftrightarrow (e)$ is a consequence of (4.8), and $(d) \Leftrightarrow (f)$ was proved in Lemma 4.2.

$(f) \Rightarrow (g)$ Let a be an invertible element in the commutative C*-algebra $E_a \cong C_0(\mathrm{Sp}(a))$. Let us denote the commutative product of $C_0(\mathrm{Sp}(a))$ by juxtaposition and the involution by *. By assumptions, we can find $b \in E_a$ such that $ab^* = b^*a = r(a)$, where $r(a)$ is the range tripotent of a, which lies in E_a, and the unit of E_a. In this case $E(a) = E_2(r(a))$. It is clear that the restriction of the product of E to E_a is given by the identity $\{x, y, z\} = xy^*z = zy^*x$ $(x, y, z \in E_a)$. We shall now check that b satisfies the desired statement. It is easy to check from the above properties that $Q(a)(b) = ab^*a = r(a)a = a$ and $Q(b)(a) = ba^*b = br(a) = b$.

Now, for each $x \in E$ we have $Q(a)Q(b)(Q(a)(x)) = Q(a)(x)$, and since $E(a) = \overline{Q(a)(E)}$, we can deduce that $Q(a)Q(b)(z) = z$ for all $z \in E(a) = E_2(r(a))$. By the Jordan identity and (4.3) we have

$$\begin{aligned} L(a, b)Q(a) &= 2Q(Q(a)(b), a) - Q(a)L(b, a) \\ &= 2Q(a) - Q(a)L(b, a) = 2Q(a) - L(a, b)Q(a), \end{aligned}$$

and thus $L(a, b)Q(a)(x) = Q(a)(x)$, for all $x \in E$, which shows by continuity of $L(a, b)$ and norm density of $Q(a)(E)$ in $E(a)$, that $L(a, b)(z) = z$ for all $z \in E_2(r(a)) = E(a)$, and by Peirce arithmetic we derive

$$L(a, b)P_2(r(a)) = P_2(r(a)) = P_2(r(a))L(a, b)P_2(r(a)). \tag{4.13}$$

On the other hand, by applying the Jordan identity, the Peirce arithmetic and (4.13), we have

$$\begin{aligned} L(b, a)Q(r(a))(x) &= 2Q(r(a))(x) - Q(r(a))L(a, b)(x) \\ &= 2Q(r(a))(x) - Q(r(a))L(a, b)P_2(r(a))(x) \\ &= 2Q(r(a))(x) - Q(r(a))(x) = Q(r(a))(x), \end{aligned}$$

for all $x \in E$, which proves that

$$L(b, a)P_2(r(a)) = P_2(r(a)) = P_2(r(a))L(b, a)P_2(r(a)). \tag{4.14}$$

$(g) \Rightarrow (h)$ Let us assume that statement (g) holds for a suitable $b \in E_2(r(a))$. In order to prove (h) we only have to show that $Q(a)$ and $Q(b)$ commute. Applying our assumptions and the Jordan identity we obtain

$$Q(a)Q(b) = 2L(a,b)L(a,b) - L(\{a,b,a\},b) = 2L(a,b)L(a,b) - L(a,b)$$
$$Q(b)Q(a) = 2L(b,a)L(b,a) - L(\{b,a,b\},a) = 2L(b,a)L(b,a) - L(b,a).$$

Since, by Peirce arithmetic, $Q(a) = Q(a)P_2(r(a))$ and $Q(b) = Q(b)P_2(r(a))$, it follows from the above identities and the assumptions that

$$Q(a)Q(b) = (2L(a,b)L(a,b) - L(a,b))P_2(r(a)) = P_2(r(a))$$
$$= (2L(b,a)L(b,a) - L(b,a))P_2(r(a)) = Q(b)Q(a).$$

$(h) \Rightarrow (i)$ If $Q(a)(b) = a$, $Q(b)(a) = b$ and $Q(a)Q(b) = Q(b)Q(a)$. Then $Q(a)Q(b) = Q(b)Q(a)$ is an idempotent in the Banach algebra $B(E)$. Let us observe that $Q(a)(E(a)) \subseteq E(a)$, and

$$Q(b)(E(a)) = Q(b)Q(a)Q(b)(E(a)) = Q(a)Q(b)Q(b)(E(a)) \subseteq E(a).$$

Therefore $Q(a)|_{E(a)}, Q(b)|_{E(a)} \in B_{\mathbb{R}}(E(a))$, where the latter symbol denotes the space of all real linear operators on $E(a)$. Moreover, since for each $x \in E$ we have $Q(a)Q(b)(Q(a)(x)) = Q(a)(x)$, we deduce, as in previous cases, that

$$Q(a)Q(b)(z) = z,$$

for all $z \in E(a)$. This means that $Q(a)|_{E(a)}$ and $Q(b)|_{E(a)}$ are invertible maps in $B_{\mathbb{R}}(E(a))$ with $Q(a)|_{E(a)}^{-1} = Q(b)|_{E(a)}$. By the bijectivity of $Q(a)|_{E(a)}$ we can find $c \in E(a)$ such that $Q(a)(c) = a$, and then

$$Q(a)^2Q(b)(c) = Q(a)Q(b)Q(a)(c) = Q(a)(c) = a,$$

which shows that a is strongly regular.

$(i) \Rightarrow (a)$ is clear. □

Let a be a regular element in a JB*-triple E. By Theorem 4.3$(a) \Leftrightarrow (f)$ the range tripotent $r(a)$ lies in E and a is an invertible element in the commutative C*-algebra E_a. The unique inverse of a in E_a is called the *generalized inverse* of a, and it will be denoted by $a^\ddagger$. Clearly, every tripotent e in E is regular and $e^\ddagger = e$. We have shown in the proof of Theorem 4.3 that

$$L(a,a^\ddagger)P_2(r(a)) = P_2(r(a)) = L(a^\ddagger,a)P_2(r(a)), \tag{4.15}$$

and

$$Q(a)Q(a^\ddagger) = P_2(r(a)) = Q(a^\ddagger)Q(a). \tag{4.16}$$

In a paper ahead of time, L.A. Harris established that statements (a) and (d) in Theorem 4.3 are equivalent in the case of J*-algebras (see [32, Lemma 3.8]).

A triple version of Proposition 2.5 reads as follows.

Proposition 4.4. *Let a be a regular element in a JB*-triple E. Suppose b is an element in E with $Q(a)(b) = a$. Then $P_2(r(a))(b) = a^\ddagger$.*

Proof. The condition $Q(a)b = a$ implies $P_2(r(a))(b) = Q(a^\ddagger)Q(a)(b) = Q(a^\ddagger)(a) = a^\ddagger$, which proves the desired conclusion. □

Remark 4.5. Let a be a regular element in a JB*-triple E. Suppose b is an element in E such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $Q(a)Q(b) = Q(b)Q(a)$. Following the arguments in the proof of Theorem 4.3$(h) \Rightarrow (i)$ we conclude that $Q(a)(E(a)) \subseteq E(a)$, $Q(b)(E(a)) \subseteq E(a)$, and $Q(a)|_{E(a)}, Q(b)|_{E(a)}$ are invertible operators in $B_{\mathbb{R}}(E(a))$ with $Q(a)|_{E(a)}^{-1} = Q(b)|_{E(a)}$. Since $b = Q(b)(a) \in E(a)$, we can easily deduce from Proposition 4.4 that $b = P_2(r(a))(b) = a^{\ddagger}$. Consequently, the unique element $b \in E$ satisfying the conditions in Theorem 4.3(h) is $a^{\ddagger}$.

Remark 4.6. [15, §4] Every C*-algebra A can be regarded as a JB*-triple with product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. We shall see next that the notion of regularity in A as a C*-algebra is equivalent to the notion of regularity in A as a JB*-triple. Namely, suppose $a \in A$ is regular in the C*-sense. Then there exists $b \in A$ satisfying $aba = a$. Then $Q(a)(b^*) = \{a, b^*, a\} = a$, showing that a is regular in the triple sense. If a is regular in the triple sense, there exists c in A such that $a = Q(a)(c) = ac^*a$ witnessing that a is regular in the C*-sense.

For a regular element $a \in A$ it seems necessary to clarify the connection between the Moore–Penrose inverse of a in the C*-algebra sense and the generalized inverse of a in the JB*-triple sense. Suppose $a^{\dagger}$ and $a^{\ddagger}$ are the Moore–Penrose inverse and the generalized inverse of a, respectively. The elements $aa^{\dagger}$ and $a^{\dagger}a$ are projections in A with $a = aa^{\dagger}a$ and $a^{\dagger}aa^{\dagger} = a^{\dagger}$. In our setting we have $Q(a)((a^{\dagger})^*) = a$, $Q((a^{\dagger})^*)(a) = (a^{\dagger})^*$ and

$$Q(a)Q((a^{\dagger})^*)(x) = a(a^{\dagger})x(a^{\dagger})a = (aa^{\dagger})x(a^{\dagger}a)$$
$$= (aa^{\dagger})^*x(a^{\dagger}a)^* = (a^{\dagger})^*a^*xa^*(a^{\dagger})^* = Q((a^{\dagger})^*)Q(a)(x).$$

Therefore $(a^{\dagger})^*$ satisfies the hypothesis in Theorem 4.3(i). Remark 4.5 proves that $(a^{\dagger})^* = a^{\ddagger}$.

We can afford now a first characterization of partial isometries in JB*-triples in terms of regularity and norm. The next result is a JB*-triple analogue of Theorem 2.6.

Theorem 4.7. [15, Corollary 3.6] *Let e be a norm-one element in a JB*-triple E. Then the following statements are equivalent:*

(a) *e is a tripotent;*
(b) *e is regular and $\|e^{\ddagger}\| \leq 1$;*
(c) *There exists b in E with $\|b\| \leq 1$ and $Q(e)(b) = e$.*

Proof. The implications $(a) \Rightarrow (b)$ follows from the fact that a non-zero tripotent tripotent e in E is regular with $e^{\ddagger} = e$. The implication $(b) \Rightarrow (c)$ is trivial. Let us prove $(c) \Rightarrow (a)$. Suppose there exists $b \in A$ with $\|b\| \leq 1$ and $Q(e)(b) = e$. Applying Proposition 4.4 we obtain $P_2(r(a))(b) = e^{\ddagger}$. By hypothesis

$$1 \geq \|b\| \geq \|P_2(r(a))(b)\| = \|e^{\ddagger}\| \geq \|e\|^{-1} = 1.$$

Since $e^{\ddagger}$ is the inverse of e in E_e, with $\|e\| = 1 = \|e^{\ddagger}\|$, we can easily deduce from the Gelfand representation of E_e that $e = e^{\ddagger}$ is a partial isometry. $\square$

We deal now with the notion of quadratic-conorm in a JB*-triple E. The *quadratic-conorm*, $\gamma^q(a)$, of an element a in a JB*-triple E was introduced by

M.J. Burgos, A. Kaidi, A. Morales, M.I. Ramírez and the second author of this survey in [15], the concrete definition reads as follows:

$$\gamma^q : E \to \mathbb{R}_0^+, \quad \gamma^q(a) = \gamma(Q(a)), \tag{4.17}$$

where $\gamma(Q(a))$ is the reduced minimum modulus of $Q(a)$ (see (2.3)).

The reciprocal statement of that in Proposition 4.1 was established in [15].

Proposition 4.8. [15, Theorem 2.3] *Let a be an element in a JB*-triple E. Then a is regular if, and only if, $Q(a)$ has norm closed range.*

Proof. The "only if" implication is given by Proposition 4.1. For the "if" implication let us assume that $Q(a)(E)$ is closed, and hence $E(a) = Q(a)(E)$. Since $a \in E_a \subseteq E(a) = Q(a)(E)$, there exists $a \in E$ such that $Q(a)(b) = a$, which concludes the argument. □

As in the case of C*-algebras, Proposition 4.8 is the first step to deal with the quadratic-conorm. By the just quoted proposition and the properties of the reduced minimum modulus, an element a in a JB*-triple E is regular if, and only, if $Q(a)$ has closed range if, and only if, $\gamma^q(a) = \gamma(Q(a)) > 0$.

We can give a precise description of the quadratic-conorm in terms of the triple spectrum.

Theorem 4.9. [15, Theorem 3.4 and Corollary 3.5] *The identity*

$$\gamma^q(a) = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\} = \inf\{s : s \in \sigma_{B(E_a)}(L(a,a)|_{E_a})\}$$

holds for every non-zero element a in a JB-triple E. Consequently, $\|a\|^2 \geq \gamma^q(a)$, for all $a \in E$, and the quadratic-conorm of an element a does not change when computed with respect to any closed complex subtriple $F \subseteq E$ with $a \in F$. Furthermore, if a is regular then $\gamma^q(a) = \|a^\ddagger\|^{-2}$.*

Proof. The second equality follows from (4.8). We shall prove the first equality. By the comments before this theorem, a is non-regular if, and only if, $\gamma^q(a) = 0$ and $0 = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\}$ by Theorem 4.3. We can therefore assume that a is regular and hence $m = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\} = \min\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\} > 0$. In other words, $\sqrt{m} = \min(\mathrm{Sp}(a)\backslash\{0\})$.

Let $a^\ddagger$ be the generalized inverse of a in E. It is known from the properties defining $a^\ddagger$ that $\|a^\ddagger\| = \dfrac{1}{\inf\{t : t \in \mathrm{Sp}(a)\backslash\{0\}\}} = \dfrac{1}{\sqrt{m}}$. We know from Theorem 4.3 and subsequent comments that $r(a) \in E$, $Q(a)Q(a^\ddagger) = Q(a^\ddagger)Q(a) = P_2(r(a))$, $Q(a)(E(a)) \subseteq E(a) = E_2(r(a))$, $Q(a^\ddagger)(E(a)) \subseteq E(a)$, $Q(a)|_{E(a)}$ and $Q(a^\ddagger)|_{E(a)}$ are invertible in $B_{\mathbb{R}}(E(a))$ and $Q(a)|_{E(a)}^{-1} = Q(a^\ddagger)|_{E(a)}$. It follows from these properties and the Peirce arithmetic that $\ker(Q(a)) = E_1(r(a)) \oplus E_0(r(a))$. For each $x \in E$ with $\mathrm{dist}(x, \ker(Q(a))) \geq 1$, we have $1 \leq \|x - P_1(r(a))(x) - P_0(r(a))(x)\| = \|P_2(r(a))(x)\|$. Then the inequality

$$1 \leq \|P_2(r(a))(x)\| = \|Q(a^\ddagger)Q(a)(x)\| \leq \|Q(a^\ddagger)\|\|Q(a)(x)\|$$

proves that

$$\|Q(a)(x)\| \geq \|Q(a^\ddagger)\|^{-1} = \|a^\ddagger\|^{-2} = m,$$

for every x in the above conditions, which assures that

$$\gamma^q(a) \geq m = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\}.$$

To prove the reciprocal inequality, take $\delta > 0$ such that $2\delta < \sqrt{m}$, $\sqrt{m}+\delta < \|a\|$ and consider the JB*-subtriple $E_a \cong C(\mathrm{Sp}(a)\backslash\{0\})$, and the element $x_0 \in E_a$ defined by

$$x_0(t) = \begin{cases} \frac{\sqrt{m}+\delta-t}{\delta}, & \text{if } t \in [\sqrt{m}, \sqrt{m}+\delta] \cap \mathrm{Sp}(a), \\ 0, & \text{if } t \in [\sqrt{m}+\delta, \infty) \cap \mathrm{Sp}(a). \end{cases}$$

Clearly $x_0 \in E_a \subseteq E(a) = E_2(r(a))$. For each $k \in \ker(Q(a)) = E_1(r(a)) \oplus E_0(r(a))$ we have $\|x_0 + k\| \geq \|P_2(r(a))(x_0 + k)\| = \|P_2(r(a))(x_0)\| = \|x_0\| = 1$, which assures that $\mathrm{dist}(x_0, \ker(Q(a))) \geq 1$. Since $\|Q(a)(x_0)\| = m$, we deduce that

$$\gamma^q(a) \leq m = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\}.$$

The final statement follows from the fact that $\mathrm{Sp}_F(a) = \mathrm{Sp}_E(a)$. □

The quadratic-conorm can be now applied to characterize tripotents.

Theorem 4.10. [15, Corollary 3.6] *Let e be an element in a JB*-triple E. The following conditions are equivalent:*

(a) $\gamma^q(e) = \|e\| = 1$;
(b) e is a non-zero tripotent.

Proof. $(a) \Rightarrow (b)$ It follows from our assumptions and Theorems 4.3 and 4.9 that e is regular. Furthermore, Theorem 4.9 also proves that $\|e\| = \gamma^q(e) = \|e^\ddagger\|^{-2} = 1$, and thus e is a tripotent by Theorem 4.7.

The implication $(b) \Rightarrow (c)$ is now clear. □

Concerning the continuity of the quadratic-conorm, having in mind the upper semi-continuity of the reduced minimum modulus [52, Corollary 10.15]), we can easily deduce that the quadratic-conorm in a JB*-triple is upper semi-continuous on $E\backslash\{0\}$ (see [15, Theorem 3.13]).

The properties defining the class of extremally rich C*-algebras, introduced by L.G. Brown and G.K. Pedersen in [12], have been recently studied in the setting of JB*-triples by F.B. Jamjoom, A. Siddiqui, H.M. Tahlawi, and the second author of this note (see [39]). Among other examples, we know that every JBW*-triple is extremally rich, and since every extremally rich C*-algebra is an extremally rich JB*-triple, the class of extremally rich JB*-triples is strictly wider than the class of JBW*-triples. A JB*-triple version of Proposition 2.12 reads as follows.

Theorem 4.11. [39, Corollary 3.7] *Let E be an extremally rich JB*-triple. Then the quadratic-conorm $\gamma^q(.)$ is continuous at a point a in E if, and only if, either a is not regular (i.e. $\gamma^q(a) = 0$) or a is Brown-Pedersen quasi-invertible (i.e. a is regular and its range tripotent lies in E and is an extreme point of its closed unit ball).* □

Since every C*-algebra A can be also regarded as an element in the class of JB*-triples, the notions of conorm and quadratic-conorm coexist for C*-algebras.

Let us clarify the relation between these functions. Let a be an element in a C*-algebra A. Suppose $a = v|a|$ is the polar decomposition of a in the von Neumann algebra A^{**}. Clearly v is a tripotent in A^{**}. When A is regarded as a JB*-triple we can also consider the range tripotent of a in A^{**}. Since $a^{[3]} = \{a, a, a\} = aa^*a = v|a||a|v^*v|a| = v|a|^3$, we can inductively prove that

$$a^{[2n+1]} = \{a, a, a^{[2n-1]}\} = v|a|^{(2n+1)}, \tag{4.18}$$

for all n in $\mathbb{N}$, and thus

$$p_t(a) = vp(|a|), \tag{4.19}$$

for every odd polynomial $p(\zeta) = \alpha_1\zeta + \alpha_3\zeta^3 + \ldots + \alpha_{2n+1}\zeta^{(2n+1)}$, where $p(|a|)$ denotes the usual continuous functional calculus at $|a|$, and $p_t(a)$ the continuous triple functional calculus at a. To avoid confusion let E denote the C*-algebra A regarded as a JB*-triple with respect to the triple product given by (4.4). As before let E_a and $E_{|a|}$ denote the JB*-subtriples of E generated by a and $|a|$, respectively. Let $\mathbb{P}^t(a)$ and $\mathbb{P}^t(|a|)$ denote the (non-necessarily closed) subtriples of all odd polynomials on a and $|a|$, respectively. Since $|a|^{[2n-1]} = |a|^{(2n-1)}$ for every natural number n, we deduce from (4.18) and (4.19) that the mapping

$$\Phi : \mathbb{P}^t(|a|) \longrightarrow \mathbb{P}^t(a), \quad p(|a|) \mapsto vp(|a|) = p_t(a)$$

is a continuous linear bijection preserving triple products. Since $\mathbb{P}^t(a) \subset E_a$ and $\mathbb{P}^t(|a|) \subset E_{|a|}$, we deduce that Φ is an isometry (compare the arguments in page 96). Therefore, by the norm density of $\mathbb{P}^t(|a|)$ and $\mathbb{P}^t(a)$ in $E_{|a|}$ and E_a, respectively, Φ extends to a surjective isometric triple isomorphism from $E_{|a|}$ onto E_a.

It is not hard to see from the positivity of $|a|$ that the subtriple $E_{|a|} \cong C_0(\mathrm{Sp}(|a|))$ of E generated by $|a|$ coincides with the C*-subalgebra $A_{|a|} \cong C_0(\sigma(|a|))$ of A generated by $|a|$. Indeed, the inclusion $E_{|a|} \subseteq A_{|a|}$ is clear because $|a| \in A_{|a|}$ and the latter is a JB*-subtriple of E. On the other hand, given two functions $0 \leq f, g \in E_{|a|} \subseteq A_{|a|} \cong C_0(\sigma(|a|))$, the sequence $(\{f, t^{[\frac{1}{2n+1}]}, g\})_n \subseteq E_{|a|}$ is monotone and converges pointwise to fg in $A_{|a|}$. Thus, Dini's theorem implies that $fg \in E_{|a|}$. Therefore, $|a|^2$ and all its powers are in $E_{|a|}$. We conclude that $E_{|a|} = A_{|a|}$ as affirmed.

The arguments in the above paragraphs show that

$$\Phi : E_{|a|} = A_{|a|} \cong C_0(\sigma(|a|)) \longrightarrow E_a \cong C_0(\mathrm{Sp}(a)),$$

$$f(|a|) \mapsto vf(|a|) = f_t(a) \tag{4.20}$$

is a surjective JB*-triple isomorphism, and in particular $v = r(a)$ and

$$\mathrm{Sp}(a) = \mathrm{Sp}(|a|) = \sigma(|a|) \cup \{0\}. \tag{4.21}$$

We recall that for each element a in a C*-algebra A, the quadratic mapping $U_a : A \to A$ is defined by $U_a(x) := axa$ ($x \in A$). It can be easily checked that $Q(a)(x) = U_a(x^*)$ for all $x \in A$.

Corollary 4.12. [15, Corollaries 4.1 and 4.2] *Let a be an element in a C*-algebra A, then the following statements hold:*

(a) $\gamma^q(a) = \gamma(|a|)^2 = \gamma(a^*a) = \gamma(aa^*)$;
(b) $\gamma^q(a) = \gamma(U_a)$.

Proof. Statement (a) is a straightforward consequence of Theorems 2.9 and 4.9 and (4.21).

(b) Let us note that $Q(a)(x^*) = U_a(x)$ for all $x \in A$. Then the equality $\gamma^q(a) = \gamma(Q(a)) = \gamma(U_a)$ follows from the definitions. □

4.1. **Real JB*-triples.** Real JB*-triples became an object of interest for Functional Analysts during the nineties as a Jordan analogue of real C*-algebras. The formal definition appears in a paper by J.M. Isidro, W. Kaup and A. Rodríguez in 1995 (see [37]). Following the just quoted article, a real Banach space is called a *real JB*-triple* if it is a norm-closed real subtriple of a complex JB*-triple. The algebraic complexification $E_c := E \oplus iE$ can be naturally equipped with a triple product extending the triple product of E and making E_c a complex Jordan Banach triple system. Real JB*-triples are naturally linked to real forms of (complex) JB*-triples. More concretely, by [37, Proposition 2.2], given a real JB*-triple E, then there exists a unique complex JB*-triple structure on E_c with respect to a norm extending the original norm on E. Furthermore, the mapping $\tau : E_c \to E_c$, $\tau(x+iy) = x - iy$ $(x, y \in E)$ is a conjugation (conjugate linear isometry of period 2) preserving triple products on E_c such that

$$E = E_c^\tau = \{z \in E_c : \tau(z) = z\}.$$

Real C*-algebras, JB-algebras, real Hilbert spaces and (complex) JB*-triples are examples of real JB*-triples. The class of real JB*-triples also includes the so-called J*B-algebras by K. Alvermann in [3].

As in the complex case, a *real JBW*-triple* is a real JB*-triple whose underlying Banach space is a dual space. It follows from [37, Lemma 4.2 and Theorem 4.4] that the bidual, E^{**}, of a real JB*-triple E is a real JBW*-triple with respect to a triple product extending the triple product of E. Every real JBW*-triple has a unique predual and its triple product is separately weak*-continuous (see [50]).

Real von Neumann algebras and JBW*-triples are examples of real JBW*-triples.

Along this subsection, E will denote a real JB*-triple with complexification E_c, and τ will stand for the conjugation on E_c satisfying $E_c^\tau = E$.

An element e in E is said to be a *tripotent* if $\{e, e, e\} = e$. Let If $\mathcal{U}(E)$ (respectively, $\mathcal{U}(E_c)$) denotes the set of all tripotents in E (respectively, the set of all tripotents in E_c) it follows from the properties of the conjugation τ that

$$\mathcal{U}(E) = \mathcal{U}(E_c)^\tau = \{e \in \mathcal{U}(E_c) : \tau(e) = e\}. \tag{4.22}$$

Let us notice that the equality in (4.22) was already observed in [37, 50, 24, 22]. Every tripotent e in E induces a Peirce decomposition of E and also of E_c in the sense we recalled in page 96. We observe that, according to the previous comments $(E_c)_j(e) \cap E = ((E_c)_j(e))^\tau = E_j(e)$ for every $j = 0, 1, 2$, and the Peirce projections of E associated with e are precisely the restrictions to E of the corresponding Peirce projections of E_c associated with e. Additional information about inner ideals on real JB*-triples can be found in [22].

An element a in E is called *regular* (respectively, *strongly regular*) if there exists b in E such that $Q(a)(b) = a$ (respectively, $Q(a)^2(b) = a$). Since τ is a conjugation and preserves triple products we can easily prove, via Theorem 4.3, that the following statements are equivalent for an element a in E.

$(R.1)$ a is regular in E;
$(R.2)$ a is regular in E_c;
$(R.3)$ a is strongly regular in E_c;
$(R.4)$ There exists an element b in E_c satisfying the following properties $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)\,Q(b) - Q(b)\,Q(a) = 0$;
$(R.5)$ a is strongly regular in E.

The implications $(R.1) \Rightarrow (R.2)$, $(R.5) \Rightarrow (R.1)$ and $(R.5) \Rightarrow (R.3)$ are clear. The unique implication which is not straightforwardly derived from Theorem 4.3 and the structure theory revised in this subsection is $(R.3) \Rightarrow (R.5)$. To see this implication, suppose a is strongly regular in E_c. The there exists $b \in E_c$ such that $Q(a)^2(b) = a$. Since τ preserves triple products and $\tau(a) = \tau$, we can easily deduce that $Q(a)^2(\tau(b)) = a$ and hence $Q(a)^2(\frac{b+\tau(b)}{2}) = a$. Since $\frac{b+\tau(b)}{2}$ lies in E, the element a is strongly regular in E.

For each element a in E the symbol $\mathbb{P}^t_{\mathbb{R}}(a)$ will denote the subspace of E of all elements of the form $p_t(a) = \alpha_1 a^{[1]} + \alpha_3 a^{[3]} + \ldots + \alpha_{2n+1} a^{[2n+1]}$, where $p(\zeta) = \alpha_1\zeta + \alpha_3\zeta^3 + \ldots + \alpha_{2n+1}\zeta^{(2n+1)}$ is an odd polynomial with real coefficients, $a^{[1]} = a$, $a^{[3]} = \{a, a, a\}$ and $a^{[2n+1]} = \{a, a, a^{[2n-1]}\}$ for $n \geq 1$, while $\mathbb{P}^t_{\mathbb{C}}(a) = \mathbb{P}^t(a)$ will have its usual meaning given in this section when a is regarded as an element in E_c. Clearly the norm closure of $\mathbb{P}^t_{\mathbb{R}}(a)$ (respectively, of $\mathbb{P}^t_{\mathbb{C}}(a)$) coincides with the real JB*-subtriple E_a of E generated by a (respectively, with the JB*-subtriple $(E_c)_a$ of E_c generated by a). By construction $(\mathbb{P}^t_{\mathbb{C}}(a))^{\tau} = \mathbb{P}^t_{\mathbb{R}}(a)$ and $(E_c)^{\tau}_a = E_a$. It follows that E_a is a real form the commutative C*-algebra $C_0(\mathrm{Sp}_{E_c}(a))$. Furthermore, as shown in [16, pages 69-70], E_a is JB*-triple isomorphic to the commutative real C*-algebra $C_0(\mathrm{Sp}_{E_c}(a), \mathbb{R})$ of all real-valued continuous functions on $\mathrm{Sp}_{E_c}(a)$ vanishing at 0. Namely, by the Stone–Weierstrass theorem there exists a homeomorphism $\sigma : \mathrm{Sp}(a) \to \mathrm{Sp}(a)$ and a unitary element $u \in C(\mathrm{Sp}(a))$ such that $\sigma(0) = 0$, $\sigma^2 = Id_{\mathrm{Sp}(a)}$, and

$$\tau(f)(t) = u(t)\overline{f(\sigma(t))},$$

for every $t \in \mathrm{Sp}(a)$ and every $f \in C_0(\mathrm{Sp}(a))$. Since in the identification $(E_c)_a \cong C_0(\mathrm{Sp}(a))$ the element a correspond to the function $t \mapsto t$ and $a = \tau(a)$, we deduce that $\sigma(t) = t$ and $u(t) = 1$ for all $t \in \mathrm{Sp}(a)$. This proves that $E_a \cong C_0(\mathrm{Sp}(a), \mathbb{R})$ is the self-adjoint part of $C_0(\mathrm{Sp}(a))$.

As in the setting of real C*-algebras, for each element a in a real JB*-triple E, we convey to set $\mathrm{Sp}(a) = \mathrm{Sp}_E(a) = \mathrm{Sp}_{E_c}(a) \subseteq [0, \|a\|]$.

The bitranspose of τ, $\tau^{**} : E^{**}_c \to E^{**}_c$, is again a conjugation on E^{**}_c and $E^{**} = (E^{**}_c)^{\tau^{**}}$ (see [37, Lemma 4.2]). It can be shown that the range tripotent in E^{**}_c, of an element a in E, is τ^{**}-symmetric, and hence lies in E^{**}. These two elements will be called the *range tripotent* and the *support tripotent* of a in E^{**}, respectively. Making use of this machinery, we can mimic the arguments in Theorem 4.3 to show that statements $(R.1)-(R.5)$ in page 108 are also equivalent to the following:

$(R.6)$ There exists an element b in E satisfying the following properties $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)\,Q(b) - Q(b)\,Q(a) = 0$;

$(R.7)$ The cubic root of a in E_a is strongly regular in E;

$(R.8)$ 0 is isolated in the triple spectrum of the cubic root of a in E_a;

$(R.9)$ 0 is isolated in $\mathrm{Sp}(a)$;

$(R.10)$ The range tripotent $r(a)$ lies in E and a is an invertible element in the commutative real C*-algebra $E_a \cong C_0(\mathrm{Sp}(a), \mathbb{R})$.

Concerning the generalized inverse, we simply observe that $a \in E$ is regular if, and only if, a is regular in E_c, and in such a case $\tau(a^\ddagger) = a^\ddagger \in E$ is called the generalized inverse of a in E. Indeed, we just observe that, by the Stone–Weierstrass theorem, for a regular element a in E, its unique generalized inverse $a^\ddagger$ (associated with the continuous functional calculus at a for the function $f(t) = \frac{1}{t}$ with $t \in \mathrm{Sp}(a)$) can be approximated in norm by odd polynomials with real coefficients in a.

A real version of Theorem 4.7 can be now stated.

Theorem 4.13. *Let e be a norm-one element in a real JB*-triple E. Then the following statements are equivalent:*

(a) *e is a tripotent;*

(b) *e is regular and $\|e^\ddagger\| \leq 1$;*

(c) *There exists b in E with $\|b\| \leq 1$ and $Q(e)(b) = e$.* □

The notion of quadratic conorm given in (4.17) also makes sense in the setting of real JB*-triples. For an element a in E the maps $Q(a)|_E : E \to E$ and $Q(a) : E_c \to E_c$ define the quadratic-conorms $\gamma^q_E(a) = \gamma(Q(a)|_E)$ and $\gamma^q_{E_c}(a) = \gamma(Q(a))$. We shall see that these quadratic-conorms coincide for every $a \in E$.

It is easy to check that $\ker(Q(a)) = \{x + iy : x, y \in \ker(Q(a)|_E)\}$, or in other words, $\ker(Q(a))$ is τ-invariant and $\ker(Q(a)|_E) = \ker(Q(a))^\tau$.

Suppose $z \in E \subset E_c$ satisfies $\mathrm{dist}(z, \ker(Q(a)|_E)) \geq 1$. The projection $P(x) = \frac{x+\tau(x)}{2}$ is contractive. Thus, for each $x + iy \in \ker(Q(a))$ we have $\|z + x + iy\| \geq \|z + x\| \geq 1$ because $x \in \ker(Q(a)|_E)$ and $\mathrm{dist}(z, \ker(Q(a)|_E)) \geq 1$. This implies that

$$\gamma^q_{E_c}(a) = \gamma(Q(a)) \leq \gamma^q_E(a) = \gamma(Q(a)|_E). \tag{4.23}$$

On the other hand, a is not regular in E if, and only if, it is not regular in E_c if, and only if, $\gamma^q_{E_c}(a) = 0$ if, and only if, $\gamma^q_E(a) = 0$. Let us assume that a is regular and $\sqrt{m} = \min\{t : t \in \mathrm{Sp}(a)\backslash\{0\}\}$. We have shown in the proof of Theorem 4.9 that the element $x_0 \in (E_c)_a \cong C_0(\mathrm{Sp}(a))$ defined by

$$x_0(t) = \begin{cases} \frac{\sqrt{m}+\delta-t}{\delta}, & \text{if } t \in [\sqrt{m}, \sqrt{m}+\delta] \cap \mathrm{Sp}(a), \\ 0, & \text{if } t \in [\sqrt{m}+\delta, \infty) \cap \mathrm{Sp}(a) \end{cases}$$

satisfies $\operatorname{dist}(x_0, \ker(Q(a))) \geq 1$ and $\|Q(a)(x_0)\| = \gamma^q_{E_c}(a)$. Since x_0 is a real-valued function, we deduce that $x_0 \in C_0(\mathrm{Sp}(a), \mathbb{R}) \cong E_a$ and hence

$$\gamma^q_{E_c}(a) = \|Q(a)(x_0)\| \geq \gamma^q_E(a). \tag{4.24}$$

Combining (4.23) and (4.24) we get $\gamma^q_E(a) = \gamma^q_{E_c}(a) = \inf\{t^2 : t \in \mathrm{Sp}(a)\backslash\{0\}\}$, for every $a \in E$, and if a is regular then $\gamma^q(a) = \|a^\ddagger\|^{-2}$. The previous formula can be now applied to obtain the next real version of Theorem 4.10.

Theorem 4.14. [16, Corollary 4.3] *Let e be an element in a real JB^*-triple E. The following conditions are equivalent:*

(a) $\gamma^q(e) = \|e\| = 1$*;*

(b) e is a non-zero tripotent. □

We culminate this section observing that a study about regular elements and quadratic-conorms for real J*B-triples (a class containing all real JB*-triples) was conducted in [16].

5. Geometric characterization in real and complex JB*-triples. Contractive perturbations

For later purposes we shall need the next lemma due to Y. Friedman and B. Russo [26]. As we commented in Section 3, Lemma 3.1 is a C*-algebra version of the following result.

Lemma 5.1. [26, Lemma 1.6] *Let e be a tripotent in a real JB*-triple E. Suppose x is a norm-one element in E with $P_2(e)x = e$. Then $x = e + P_0(e)x$.* □

We note that the original statement of the above lemma in [26, Lemma 1.6] is only established for (complex) JB*-triples. However, the statement for real JB*-triples can be easily deduced from the original result by just regarding real JB*-triples as real forms of their complexifications.

It is well known that the unit element in an associative complex Banach algebra A is an extreme point of the closed unit ball of A (see [58, Proposition 1.6.6]). A unital JB-algebra J with unit 1 is an example of order unit space in the sense employed in [30, §1.2]. It follows from [30, Lemmas 1.2.2 and 1.2.5] that the set $S(J)$ of all states of J relative to 1 (i.e. the norm-one positive elements in J^*) is a norming set for J, that is, for every $a \in J$ there exists $\varphi \in S(J)$ with $|\varphi(a)| = \|a\|$. In particular, 1 is an extreme point of the closed unit ball of J. The self-adjoint part of a JB*-algebra (or more generally, the self-adjoint part of a unital J*B-algebra in the sense of Alvermann [3]) is a JB-algebra (see [60] and [30, §3.8]), consequently, the unit of every unital JB*-algebra (respectively, of every unital J*B-algebra) A is an extreme point in the closed unit ball of A.

In Section 3 we have revisited the geometric characterization of partial isometries in C*-algebras obtained by C.A. Akemann and N. Weaver in [2] (see Theorem 3.2). This result is an useful tool for many problems where the answer is specially tractable if instead of projections we need to preserve partial isometries. Akemann-Weaver theorem naturally leads to the question whether a geometric characterization of tripotents in general JB*-triples is affordable. This problem was studied by J. Martínez and the authors of this survey in the wider setting of tripotents in real or complex JB*-triples (see [24]). Keeping in mind the notation in Section 3 the characterization is established in the next result.

Theorem 5.2. [24, Theorems 2.1 and 2.3] *Let e be a norm-one element in a real or complex JB*-triple E. Then e is a tripotent if, and only if, $D_1^E(e) = D_2^E(e)$. Moreover, for each tripotent e in E we have $D_1^E(e) = E_0(e)$.*

Proof. We shall assume that E is a real JB*-triple.

($\Rightarrow$) Let e be a tripotent. The inclusion $D_2^E(e) \subseteq D_1^E(e)$ is always true for every norm-one element in a Banach space. Therefore, it will be enough to prove the reverse inclusion.

We observe that in a real JB*-triple E, the Peirce 2-subspace $E_2(e)$ is a unital real J*B-subalgebra of the unital JB*-algebra $(E_c)_2(e)$ associated with e in the complexification of E, and e is the unit element in both Jordan algebras. Then, given $y \in D_1^E(e)$ (i.e. $\|e \pm \alpha y\| = 1$ with $\alpha > 0$), we have $\|e \pm \alpha P_2(e)y\| \leq 1$. Since e is the unit element in the J*B-algebra $E_2(e)$ it follows that $P_2(e)(y) = 0$.

Since $P_2(e)(e + \alpha y) = e$, Lemma 5.1 assures that $e + \alpha y = e + P_0(e)(\alpha y)$, and thus $e \perp y = P_0(e)y$, which proves that y belongs to $D_2^E(e)$ (see (4.9)).

We have also shown that $D_1^E(e) = E_0(e)$ for every non-zero tripotent e in E.

($\Leftarrow$) Arguing by contradiction, we suppose that e is not a tripotent in E. Let us identify E_e with $C_0(\mathrm{Sp}(e), \mathbb{R})$ and e with the function $t \mapsto t$ ($t \in \mathrm{Sp}(e)$) (see §4.1). Since e is not a tripotent there exists $t_0 \in]0, 1[\cap \mathrm{Sp}(e)$. We define $h : \mathrm{Sp}(e) \to \mathbb{R}$ the function given by

$$h(t) = \begin{cases} \frac{1-t_0}{t_0} t, & \text{if } t \in [0, t_0] \cap \mathrm{Sp}(e), \\ 1 - t, & \text{if } t \in [t_0, 1] \cap \mathrm{Sp}(e). \end{cases}$$

and we set $y = h_t(e) \in E_e$. It is straightforward to check, via continuous triple functional calculus, that $\|e \pm y\| = \|e(t) \pm h(t)\|_{C_0(\mathrm{Sp}(e))} = 1$ (i.e. y belongs to $D_1^E(e)$). However, y does not belong to $D_2^E(e)$. Namely, if we take, for example, $\beta = \frac{1}{1-t_0}$, we have $\|e + \beta y\| = \|t + \beta h(t)\|_{C_0(\mathrm{Sp}(e))} = 1 + t_0 > 1$, $\|e\| = 1$ and $\|\beta y\| = 1$. ☐

Let x and y be two elements in a real or complex JB*-triple E. Following [49], the *Bergman operator* $B(x, y) : E \to E$ is defined by

$$B(x, y) := Id_E - 2L(x, y) + Q(x)Q(y).$$

For each tripotent e in E, $B(e, e) = P_0(e)$.

Bergman operators and orthogonal complements can be also applied to determine tripotents in JB*-triples.

Theorem 5.3. ([25, Proposition 9] *Let e be a norm-one element in a real or complex JB*-triple E. The following statements are equivalent:*

(a) e is a tripotent;
(b) $B(e, e)(E) = \{e\}^{\perp} = \{x \in E : e \perp x\}$.

Proof. As in the previos result, we may asume that E is a real JB*-triple.

$(a) \Rightarrow (b)$ Suppose e is a tripotent. In this case $B(e,e) = P_0(e)$. Since $E_0(e) = P_0(e)(E) = B(e,e)(E)$, we deduce from Peirce arithmetic that $x \perp e$ for every $x \in E_0(e)$, equivalently $B(e,e)(E) \subseteq \{e\}^{\perp}$.

On the other hand, if $x \perp e$, then $\{e,e,x\} = 0$, and hence $x \in E_0(e) = B(e,e)(E)$.

$(b) \Rightarrow (a)$ Suppose $B(e,e)(E) = \{e\}^{\perp}$. Let us observe that in the JB*-subtriple $E_e \cong C_0(\mathrm{Sp}(e),\mathbb{R})$ two elements are orthogonal if, and only if, they have disjoint supports as continuous functions. Since under this identification e corresponds to the function $t \mapsto t$ $(t \in \mathrm{Sp}(e))$, we deduce that $E_e \cap \{e\}^{\perp} = \{0\}$. By assumptions, $B(e,e)(e) = e - 2L(e,e)(e) + Q(e)^2(e) = e - 2e^{[3]} + e^{[5]} \in \{e\}^{\perp} \cap E_e$, and thus $e - 2e^{[3]} + e^{[5]} = 0$. In the identification $E_e \cong C_0(\mathrm{Sp}(e),\mathbb{R})$ the element $e - 2e^{[3]} + e^{[5]}$ corresponds to the polynomial $t - 2t^3 + t^5$, so $e - 2e^{[3]} + e^{[5]} = 0$ forces to $\mathrm{Sp}(e) \subseteq \{0,1\}$, which proves that e is a tripotent. $\square$

The above result appeared in [25, Proposition 9] where it was obtained as a consequence of a more general conclusion affirming that the identity

$$\{x\}^{\perp} = \{y \in E : B(x,x)(y) = y\}$$

holds for every element x in a JB*-triple E with $\|x\| < \sqrt{2}$ (see [25, Proposition 7]).

Kadison's theorem determining the extreme points of the closed unit ball of a C*-algebra (see Corollary 3.4) was extended to the categories of complex and real JB*-triples by W. Kaup and H. Upmeier [45, Proposition 3.5] and J.M. Isidro, W. Kaup and A. Rodríguez [37, Lemma 3.3], respectively. We can rediscover now these results as applications of the geometric characterization of tripotents.

Corollary 5.4. ([45, Proposition 3.5], [37, Lemma 3.3]) *Let e be a norm-one element in a real or complex JB*-triple E. Then the following statements are equivalent:*

(a) *e is an extreme point of the closed unit ball of E;*
(b) *$D_1^E(e) = \{0\}$;*
(c) *e is a tripotent and $E_0(e) = \{0\}$;*
(d) *$B(e,e)(E) = \{0\}$.*

Proof. $(a) \Leftrightarrow (b)$ Obviously, $D_1^E(e) = \{0\}$ if e is an extreme point of the closed unit ball of E. Reciprocally, if $D_1^E(e) = \{0\}$, and $e = \frac{1}{2}(x+y)$ with $\|x\|, \|y\| = 1$, then taking $z = \frac{1}{2}(x-y)$ we have $\|z\| \leq 1$, $\|e+z\| = \|x\| = 1$ and $\|e-z\| = \|y\| = 1$. Therefore $z \in D_1^E(e) = \{0\}$, and hence $x = y = e$.

$(b) \Rightarrow (c)$ Since the inclusions $\{0\} \subseteq D_2^E(e) \subseteq D_1^E(e)$ always hold, the assumption $D_1^E(e) = \{0\}$, implies that $D_2^E(e) = D_1^E(e) = \{0\}$. Theorem 5.2 gives (c).

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (b)$ Since $\{0\} \subseteq \{e\}^{\perp} \subseteq B(e,e)(E)$ and the later set coincides with $\{0\}$ by hypothesis, Theorem 5.3 assures that e is a tripotent, and Theorem 5.2 proves that $D_1^E(e) = E_0(e) = \{e\}^{\perp} = \{0\}$. $\square$

It should be remarked here that a generalization of the celebrated Kadison's characterization of the extreme points of the closed unit ball of a C*-algebra (see Corollary 3.4) was established by L.A. Harris in [31, Theorem 11] in what is a forerunner of the previous Corollary 5.4. A thorough survey on unitary elements in C*-algebras, JB*-algebras, and JB*-triples was conducted by A. Rodríguez in [57].

In the case of complex JB*-triples, the geometric characterization of tripotents presented in Theorem 5.2 was later rediscovered by R.V. Hügli in [36]. In the just quoted paper, Hügli also added some other equivalent reformulations. We can now derive the new reformulations as a consequence of Theorem 5.2.

Corollary 5.5. [36, Theorem 4.1] *Let e be a norm-one element in a complex JB*-triple E, and let $\mathcal{B}_E$ denote the closed unit ball of E. Then the following statements are equivalent:*

(a) *e is a tripotent;*
(b) $\{y \in E : \|e \pm y\| = 1\} \subseteq \{e\}^{\perp} \cap \mathcal{B}_E$;
(c) $\{y \in E : \|e \pm y\| = 1\} = \{y \in E : \|ie \pm y\| = 1\}$.

Proof. $(a) \Rightarrow (b)$ Suppose e is a tripotent. The set $\{y \in E : \|e \pm y\| = 1\}$ is clearly contained in $D_1^E(e)$. We observe that every element y in the first set satisfies $\|y\| \leq 1$. On the other hand, the assumptions combined with Theorem 5.2 show that $D_1^E(e) = D_2^E(e) = E_0(e) = \{e\}^{\perp}$.

$(b) \Rightarrow (a)$ Let us assume that $\{y \in E : \|e \pm y\| = 1\} \subseteq \{e\}^{\perp} \cap \mathcal{B}_E$. Given $z \in D_1^E(e)$, there exists a positive α such that $\alpha z \in \{y \in E : \|e \pm y\| = 1\}$, and hence $\alpha z \in \{e\}^{\perp} \cap \mathcal{B}_E$. Since orthogonal elements are M-orthogonal in the geometric sense (see (4.9)), we derive that $z \in D_2^E(e)$. This shows that $D_1^E(e) = D_2^E(e)$ and Theorem 5.2 gives (a).

Although [36, Theorem 4.1] only concerns with complex JB*-triples, the equivalence $(a) \Leftrightarrow (b)$ also holds for real JB*-triples. Actually statement (b) is equivalent to

(b') $\{y \in E : \|e \pm y\| = 1\} = \{e\}^{\perp} \cap \mathcal{B}_E$.

Namely, we have shown in the proof of $(b) \Rightarrow (a)$ that $\{e\}^{\perp} \cap \mathcal{B}_E \subseteq D_2^E(e)$, and thus for a tripotent e in E we have

$$\{y \in E : \|e \pm y\| = 1\} \subseteq \{e\}^{\perp} \cap \mathcal{B}_E \subseteq D_2^E(e) \cap \mathcal{B}_E = D_1^E(e) \cap \mathcal{B}_E$$
$$= E_0(e) \cap \mathcal{B}_E = (\text{see } (4.9)) = \{y \in E : \|e \pm y\| = 1\}.$$

$(a) \Rightarrow (c)$ By $(a) \Leftrightarrow (b')$ we get $\{y \in E : \|e \pm y\| = 1\} = \{e\}^{\perp} \cap \mathcal{B}_E$ and $\{y \in E : \|ie \pm y\| = 1\} = \{ie\}^{\perp} \cap \mathcal{B}_E = \{e\}^{\perp} \cap \mathcal{B}_E$, which proves (c).

$(c) \Rightarrow (a)$ As in the proof of Theorem 5.2, if e is not a tripotent in E, there exists $t_0 \in]0,1[\cap \mathrm{Sp}(e)$. We identify E_e with $C_0(\mathrm{Sp}(e))$ and e with the function

$t \mapsto t$ ($t \in \mathrm{Sp}(e)$). We define $h : \mathrm{Sp}(e) \to \mathbb{C}$ the function given by

$$h(t) = \begin{cases} i\frac{\sqrt{1-t_0^2}}{t_0}t, & \text{if } t \in [0, t_0] \cap \mathrm{Sp}(e), \\ i\sqrt{1-t^2}, & \text{if } t \in [t_0, 1] \cap \mathrm{Sp}(e) \end{cases}$$

and we set $y = h_t(e) \in E_e$. It is straightforward to check that $\|e \pm y\| = 1$ but $\|ie + y\| \geq |it_0 + h(t_0)| = \sqrt{1-t_0^2} + t_0 > 1$. $\square$

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Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

E-mail address: [1]pacopolo@ugr.es

E-mail address: [2]aperalta@ugr.es