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2-LOCAL DERIVATIONS ON MATRIX ALGEBRAS AND ALGEBRAS OF MEASURABLE OPERATORS

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Dedicated to the memory of Uffe Haagerup

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Abstract. Let $\mathcal{A}$ be a unital Banach algebra such that any Jordan derivation from $\mathcal{A}$ into any $\mathcal{A}$-bimodule $\mathcal{M}$ is a derivation. We prove that any 2-local derivation from the algebra $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ $(n \geq 3)$ is a derivation. We apply this result to show that any 2-local derivation on the algebra of locally measurable operators affiliated with a von Neumann algebra without direct abelian summands is a derivation.

1. Introduction

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ the field of complex numbers and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear map D from $\mathcal{A}$ to $\mathcal{M}$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$. If it satisfies a weaker condition $D(x^2) = D(x)x + xD(x)$ for every $x \in \mathcal{A}$ then it is called a *Jordan derivation*. It is easy to verify that each element $a \in \mathcal{M}$ implements a derivation D_a from $\mathcal{A}$ into $\mathcal{M}$ by $D_a(x) = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are called *inner derivations*.

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In 1990, Kadison [12] and Larson and Sourour [15] independently introduced the concept of local derivation. A linear map $\Delta : \mathcal{A} \to \mathcal{M}$ is called a *local derivation* if for every $x \in \mathcal{A}$ there exists a derivation D_x (depending on x) such that $\Delta(x) = D_x(x)$. It would be interesting to consider under which conditions local derivations automatically become derivations. Many partial results have been done in this problem. In [12] Kadison shows that every norm-continuous local derivation from a von Neumann algebra M into a dual M-bimodule is a derivation. In [11] Johnson extends Kadison's result and proves every local derivation from a C^*-algebra $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is a derivation.

Similar problems for local derivations on algebras of measurable operators $S(M)$ and locally measurable operators $LS(M)$, affiliated with a von Neumann algebra M, have been considered in [4] and [9]. Namely, it was proved that if M is a von Neumann algebra without abelian direct summand then every local derivation on $LS(M)$ is a derivation. Moreover, for abelian von Neumann algebras M necessary and sufficient condition are given in [5] for $S(M) = LS(M)$ to admit local derivations which are not derivations (see for details the survey [4, Section 5]).

In 1997, Šemrl [17] initiated the study of so-called 2-local derivations and 2-local automorphisms on algebras. Namely, he described such maps on the algebra $B(H)$ of all bounded linear operators on an infinite dimensional separable Hilbert space H.

In the above notations, map $\Delta : \mathcal{A} \to \mathcal{M}$ (not necessarily linear) is called a *2-local derivation* if, for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{M}$ such that $D_{x,y}(x) = \Delta(x)$ and $D_{x,y}(y) = \Delta(y)$.

Afterwards local derivations and 2-local derivations have been investigated by many authors on different algebras and many results have been obtained in [1, 2, 3, 5, 12, 14, 17].

Recall that an algebra $\mathcal{A}$ is called a regular (in the sense of von Neumann) if for each $a \in \mathcal{A}$ there exists $b \in \mathcal{A}$ such that $a = aba$. Let $M_n(\mathcal{A})$ be the algebra of all $n \times n$ matrices over a unital commutative regular algebra $\mathcal{A}$. In [5], we prove that every 2-local derivation on $M_n(\mathcal{A})$, $n \geq 2$, is a derivation. We applied this result to a description of 2-local derivations on the algebras of measurable operators $S(M)$ and locally measurable operators $LS(M)$ affiliated with a type I finite von Neumann algebra M. Further this result was extended to type I_∞ von Neumann algebras: it was proved that in this case every 2-local derivations on the algebra of locally measurable operators is a derivation (see [4, Theorem 6,7]). Moreover in [5] we also gave necessary and sufficient conditions for a commutative regular algebra, in particular for the algebra $S(M)$ of measurable operators affiliated with an abelian von Neumann algebra M, to admit 2-local derivations which are not derivations. In [3] we considered a unital semi-prime Banach algebra $\mathcal{A}$ with the inner derivation property and proved that any 2-local derivation on the algebra $M_{2^n}(\mathcal{A})$, $n \geq 2$, is a derivation. We have applied this result to AW^*-algebras and proved that any 2-local derivation on an arbitrary AW^*-algebra is a derivation. In [10], W. Huang, J. Li and W. Qian, have characterized derivations and 2-local derivations from $M_n(\mathcal{A})$ into $M_n(\mathcal{M}), n \geq 2$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$

and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule. They considered a unital Banach algebra such that any Jordan derivation from the algebra $\mathcal{A}$ into any $\mathcal{A}$-bimodule $\mathcal{M}$ is an inner derivation and proved that any 2-local derivation from the algebra $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ $(n \geq 3)$ is a derivation, when $\mathcal{A}$ is commutative and commutes with $\mathcal{M}$.

In the present paper we shall consider matrix algebras over unital (non commutative in general) Banach algebras and describe 2-local derivations from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$, where $\mathcal{A}$ is a unital Banach algebra such that any Jordan derivation from the algebra $\mathcal{A}$ into any $\mathcal{A}$-bimodule $\mathcal{M}$ is a derivation. The main result of Section 2 asserts that under the above conditions every 2-local derivation from the algebra $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ $(n \geq 3)$ is a derivation.

In Section 3, we apply the main result of the previous section to algebras of locally measurable operators affiliated with von Neumann algebras. Namely, we extend all above mentioned results from [3, 4, 5, 10] and prove that for an arbitrary von Neumann algebra M without abelian direct summands every 2-local derivation on each subalgebra $\mathcal{A}$ of the algebra $LS(M)$, such that $M \subseteq \mathcal{A}$, is a derivation. A similar result for local derivation is obtained in [9, Theorem 1] (see also [4, Theorem 5.5]).

2. 2-LOCAL DERIVATIONS ON MATRIX ALGEBRAS

If $\Delta : \mathcal{A} \to \mathcal{M}$ is a 2-local derivation, then from the definition it easily follows that Δ is homogenous. At the same time,

$$\Delta(x^2) = \Delta(x)x + x\Delta(x)$$

for each $x \in \mathcal{A}$. This means that additive (and hence, linear) 2-local derivation is a Jordan derivation.

In [8] Brešar suggested various conditions on an algebra $\mathcal{A}$ under which any Jordan derivation from $\mathcal{A}$ into any $\mathcal{A}$-bimodule $\mathcal{M}$ is a derivation.

In the present paper we shall consider algebras with the following property:

(J): *any Jordan derivation from the algebra $\mathcal{A}$ into any $\mathcal{A}$-bimodule $\mathcal{M}$ is a derivation.*

Therefore, in the case of algebras with the property **(J)** in order to prove that a 2-local derivation $\Delta : \mathcal{A} \to \mathcal{M}$ is a derivation it is sufficient to prove that $\Delta : \mathcal{A} \to \mathcal{M}$ is additive.

Throughout this paper, $\mathcal{A}$ is a unital Banach algebra over $\mathbb{C}$, $\mathcal{M}$ is an $\mathcal{A}$-bimodule with $\mathbf{1}x = x\mathbf{1} = x$ for all $x \in \mathcal{M}$, where $\mathbf{1}$ is the unit element of $\mathcal{A}$.

The following theorem is the main result of this section.

Theorem 2.1. *Let $\mathcal{A}$ be a unital Banach algebra with the property **(J)**, $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule and let $M_n(\mathcal{A})$ be the algebra of all $n \times n$-matrices over $\mathcal{A}$, where $n \geq 3$. Then any 2-local derivation Δ from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ is a derivation.*

The proof of Theorem 2.1 consists of two steps. In the first step we shall show additivity of Δ on the subalgebra of diagonal matrices from $M_n(\mathcal{A})$.

Let $\{e_{i,j}\}_{i,j=1}^n$ be the system of matrix units in $M_n(\mathcal{A})$. For $x \in M_n(\mathcal{A})$ by $x_{i,j}$ we denote the (i,j)-entry of x, where $1 \leq i, j \leq n$. We shall, if necessary, identify this element with the matrix from $M_n(\mathcal{A})$ whose (i,j)-entry is $x_{i,j}$, other entries are zero, i.e. $x_{i,j} = e_{i,i}xe_{j,j}$.

Each element $x \in M_n(\mathcal{A})$ has the form

$$x = \sum_{i,j=1}^{n} x_{ij}e_{ij}, \ x_{ij} \in \mathcal{A}, i, j \in \overline{1, n}.$$

Let $\delta : \mathcal{A} \to \mathcal{M}$ be a derivation. Setting

$$\overline{\delta}(x) = \sum_{i,j=1}^{n} \delta(x_{ij})e_{ij}, \ x_{ij} \in \mathcal{A}, i, j \in \overline{1, n} \tag{2.1}$$

we obtain a well-defined linear operator $\overline{\delta}$ from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$. Moreover $\overline{\delta}$ is a derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$.

It is known [10, Theorem 2.1] that every derivation D from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ can be represented as a sum

$$D = ad(a) + \overline{\delta}, \tag{2.2}$$

where $\mathrm{ad}(a)$ is an inner derivation implemented by an element $a \in M_n(\mathcal{M})$, while $\overline{\delta}$ is the derivation of the form (2.1) generated by a derivation δ from $\mathcal{A}$ into $\mathcal{M}$.

Consider the following two matrices:

$$u = \sum_{i=1}^{n} \frac{1}{2^i} e_{i,i}, \ v = \sum_{i=2}^{n} e_{i-1,i}. \tag{2.3}$$

It is easy to see that an element $x \in M_n(\mathcal{M})$ commutes with u if and only if it is diagonal, and if an element $a \in M_n(\mathcal{M})$ commutes with v, then a is of the form

$$a = \begin{pmatrix} a_1 & a_2 & a_3 & . & \dots & a_n \\ 0 & a_1 & a_2 & . & \dots & a_{n-1} \\ 0 & 0 & a_1 & . & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & . & a_1 & a_2 \\ 0 & 0 & \dots & . & 0 & a_1 \end{pmatrix}. \tag{2.4}$$

A result, similar to the following one, was proved in [5, Lemma 4.4] for matrix algebras over commutative regular algebras.

Further in Lemmata 2.2–2.5 we assume that $n \geq 2$.

Lemma 2.2. *For every 2-local derivation Δ from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ there exists a derivation D such that $\Delta|_{sp\{e_{i,j}\}_{i,j=1}^n} = D|_{sp\{e_{i,j}\}_{i,j=1}^n}$, where $sp\{e_{i,j}\}_{i,j=1}^n$ is the linear span of the set $\{e_{i,j}\}_{i,j=1}^n$.*

Proof. Take a derivation D from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ such that

$$\Delta(u) = D(u), \, \Delta(v) = D(v),$$

where u, v are the elements from (2.3). Replacing Δ by $\Delta - D$, if necessary, we can assume that $\Delta(u) = \Delta(v) = 0$.

Let $i, j \in \overline{1, n}$. Take a derivation $D = \mathrm{ad}(h) + \overline{\delta}$ of the form (2.2) such that

$$\Delta(e_{i,j}) = [h, e_{i,j}] + \overline{\delta}(e_{ij}),\ \Delta(u) = [h, u] + \overline{\delta}(u).$$

Since $\Delta(u) = 0$ and $\overline{\delta}(u) = 0$, it follows that $[h, u] = 0$, and therefore h has a diagonal form, i.e. $h = \sum_{s=1}^{n} h_s e_{s,s}$, $h_s \in \mathcal{A}$, $s \in \overline{1, n}$.

In the same way, but starting with the element v instead of u, we obtain

$$\Delta(e_{i,j}) = be_{i,j} - e_{i,j}b,$$

where b has the form (2.4), depending on $e_{i,j}$. So

$$\Delta(e_{i,j}) = he_{i,j} - e_{i,j}h = be_{i,j} - e_{i,j}b.$$

It follows from $he_{i,j} - e_{i,j}h = (h_i - h_j)e_{i,j}$ and $[be_{i,j} - e_{i,j}b]_{i,j} = 0$ that $\Delta(e_{i,j}) = 0$.

Now let us take a matrix $x = \sum_{i,j=1}^{n} \lambda_{i,j} e_{i,j} \in M_n(\mathbb{C})$. Then

$$\begin{aligned}
e_{i,j}\Delta(x)e_{i,j} &= e_{i,j}D_{e_{i,j},x}(x)e_{i,j}\\
&= D_{e_{i,j},x}(e_{i,j}xe_{i,j}) - D_{e_{i,j},x}(e_{i,j})xe_{i,j} - e_{i,j}xD_{e_{i,j},x}(e_{i,j})\\
&= D_{e_{i,j},x}(\lambda_{j,i}e_{i,j}) - \Delta(e_{i,j})xe_{i,j} - e_{i,j}x\Delta(e_{i,j})\\
&= \lambda_{j,i}D_{e_{i,j},x}(e_{i,j}) - 0 - 0 = \lambda_{j,i}\Delta(e_{i,j}) = 0,
\end{aligned}$$

i.e. $e_{i,j}\Delta(x)e_{i,j} = 0$ for all $i, j \in \overline{1, n}$. This means that $\Delta(x) = 0$. The proof is complete. $\square$

Further in Lemmata 2.3–2.8 we assume that Δ is a 2-local derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$ such that $\Delta|_{\mathrm{sp}\{e_{i,j}\}_{i,j=1}^{n}} = 0$.

Let $\Delta_{i,j}$ be the restriction of Δ onto $\mathcal{A}_{i,j} = e_{i,i}M_n(\mathcal{A})e_{j,j}$, where $1 \le i, j \le n$.

Lemma 2.3. *$\Delta_{i,j}$ maps $\mathcal{A}_{i,j}$ into itself.*

Proof. Let us show that

$$\Delta_{i,j}(x) = e_{i,i}\Delta(x)e_{j,j} \tag{2.5}$$

for all $x \in \mathcal{A}_{i,j}$.

Take $x = x_{i,j} \in \mathcal{A}_{i,j}$, and consider a derivation $D = \mathrm{ad}(h) + \overline{\delta}$ of the form (2.2) such that

$$\Delta(x) = [h, x] + \overline{\delta}(x),\ \Delta(u) = [h, u] + \overline{\delta}(u),$$

where u is the element from (2.3). Since $\Delta(u) = 0$ and $\overline{\delta}(u) = 0$, it follows that $[h, u] = 0$, and therefore h has a diagonal form. Then $\Delta(x) = (h_i - h_j)e_{ij} + \delta(x_{ij})e_{ij}$. This means that $\Delta(x) \in \mathcal{A}_{i,j}$. The proof is complete. $\square$

Lemma 2.4. *Let $x = \sum_{i=1}^{n} x_{i,i}$ be a diagonal matrix. Then*

$$e_{k,k}\Delta(x)e_{k,k} = \Delta(x_{k,k}) \tag{2.6}$$

for all $k \in \overline{1, n}$.

Proof. Take a derivation $D = \mathrm{ad}(a) + \overline{\delta}$ of the form (2.2) such that

$$\Delta(x) = [a, x] + \overline{\delta}(x) \text{ and } \Delta(x_{k,k}) = [a, x_{k,k}] + \overline{\delta}(x_{kk}).$$

Using equality (2.5), we obtain that

$$\Delta(x_{k,k}) = e_{k,k}\Delta(x_{k,k})e_{k,k} = e_{k,k}[a, x_{k,k}]e_{k,k} + e_{k,k}\overline{\delta}(x_{k,k})e_{k,k} = [a_{k,k}, x_{k,k}] + \delta(x_{k,k}).$$

Since x is a diagonal matrix, we get

$$e_{k,k}\Delta(x)e_{k,k} = e_{k,k}[a, x]e_{k,k} + e_{kk}\overline{\delta}(x)e_{k,k} = [a_{k,k}, x_{k,k}] + \delta(x_{k,k}).$$

Thus $e_{k,k}\Delta(x)e_{k,k} = \Delta(x_{k,k})$. The proof is complete. □

Lemma 2.5. *Let $x = x_{i,i} \in \mathcal{A}_{i,i}$. Then*

$$e_{j,i}\Delta(x)e_{i,j} = \Delta(e_{j,i}xe_{i,j}) \tag{2.7}$$

for every $j \in \{1, \cdots, n\}$.

Proof. For $i = j$ we have already proved (see Lemma 2.4).

Suppose that $i \neq j$. For an arbitrary element $x = x_{i,i} \in \mathcal{A}_{i,i}$, consider $y = x + e_{j,i}xe_{i,j} \in \mathcal{A}_{i,i} + \mathcal{A}_{j,j}$. Take a derivation $D = \mathrm{ad}(a) + \overline{\delta}$ such that

$$\Delta(y) = [a, y] + \overline{\delta}(y) \text{ and } \Delta(v) = [a, v] + \overline{\delta}(v),$$

where v is the element from (2.3). Since $\Delta(v) = 0$ and $\overline{\delta}(v) = 0$, it follows that a has the form (2.4). By Lemma 2.4 we obtain that

$$\begin{aligned}
e_{j,i}\Delta(x)e_{i,j} &= e_{j,i}e_{i,i}\Delta(y)e_{i,i}e_{i,j} = e_{j,i}[a, y]e_{i,j} + e_{j,i}\overline{\delta}(y)e_{i,j} \\
&= ([a_1, x] + \delta(x))\, e_{j,j}, \\
\Delta(e_{j,i}xe_{i,j}) &= e_{j,j}\Delta(y)e_{j,j} = e_{j,j}[a, y]e_{j,j} + e_{j,j}\overline{\delta}(y)e_{j,j} \\
&= e_{j,j}[a, x + e_{j,i}xe_{i,j}]e_{j,j} + e_{j,j}\delta(x)e_{j,j} = ([a_1, x] + \delta(x))\, e_{j,j}.
\end{aligned}$$

The proof is complete. □

Further in Lemmata 2.6–2.13 we assume that $n \geq 3$.

Lemma 2.6. *$\Delta_{i,i}$ is additive for all $i \in \overline{1, n}$.*

Proof. Let $i \in \overline{1, n}$. Since $n \geq 3$, we can take different numbers k, s such that $(k - i)(s - i) \neq 0$.

For arbitrary $x, y \in \mathcal{A}_{i,i}$ consider the diagonal element $z \in \mathcal{A}_{i,i} + \mathcal{A}_{k,k} + \mathcal{A}_{s,s}$ such that $z_{i,i} = x + y$, $z_{k,k} = x$, $z_{s,s} = y$. Take a derivation $D = \mathrm{ad}(a) + \overline{\delta}$ such that

$$\Delta(z) = [a, z] + \overline{\delta}(z) \text{ and } \Delta(v) = [a, v] + \overline{\delta}(v),$$

where v is the element from (2.3). Since $\Delta(v) = 0$ and $\overline{\delta}(v) = 0$, it follows that a has the form (2.4). Using Lemmata 2.4 and 2.5 we obtain that

$$\begin{aligned}
\Delta_{i,i}(x + y) &\overset{(2.6)}{=} e_{i,i}\Delta(z)e_{i,i} = e_{i,i}[a, z]e_{i,i} + e_{i,i}\overline{\delta}(z)e_{i,i} \\
&= ([a_1, x + y] + \delta(x + y))\, e_{i,i}, \\
\Delta_{i,i}(x) &\overset{(2.7)}{=} e_{i,k}\Delta(e_{k,i}xe_{i,k})e_{k,i} \overset{(2.6)}{=} e_{i,k}e_{k,k}\Delta(z)e_{k,k}e_{k,i} \\
&= e_{i,k}[a, z]e_{k,i} + e_{i,k}\overline{\delta}(z)e_{k,i} = ([a_1, x] + \delta(x))\, e_{i,i}, \\
\Delta_{i,i}(y) &\overset{(2.7)}{=} e_{i,s}\Delta(e_{s,i}ye_{i,s})e_{s,i} \overset{(2.6)}{=} e_{i,s}e_{s,s}\Delta(z)e_{s,s}e_{s,i} \\
&= e_{i,s}[a, z]e_{s,i} + e_{i,s}\overline{\delta}(z)e_{s,i} = ([a_1, y] + \delta(y))\, e_{i,i}.
\end{aligned}$$

Hence
$$\Delta_{i,i}(x+y) = \Delta_{i,i}(x) + \Delta_{i,i}(y).$$
The proof is complete. $\square$

As it was mentioned in the beginning of the section any additive 2-local derivation is a Jordan derivation. Since $\mathcal{A}_{i,i} \cong \mathcal{A}$ has the property **(J)**, Lemma 2.6 implies the following result.

Lemma 2.7. *$\Delta_{i,i}$ is a derivation for all $i \in \overline{1,n}$.*

Denote by $\mathcal{D}_n(\mathcal{A})$ the set of all diagonal matrices from $M_n(\mathcal{A})$, i.e. the set of all matrices of the following form
$$x = \left(\begin{array}{ccccc} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & x_{n-1} & 0 \\ 0 & 0 & \dots & 0 & x_n \end{array}\right).$$

Let us consider a derivation $\overline{\Delta_{1,1}}$ of the form (2.1). By Lemmata 2.4 and 2.5 we obtain that

Lemma 2.8. $\Delta|_{\mathcal{D}_n(\mathcal{A})} = \overline{\Delta_{1,1}}|_{\mathcal{D}_n(\mathcal{A})}$ *and* $\overline{\Delta_{1,1}}|_{sp\{e_{i,j}\}_{i,j=1}^n} = 0$.

Now we are in position to pass to the second step of our proof. In this step we show that if a 2-local derivation Δ satisfies the following conditions
$$\Delta|_{\mathcal{D}_n(\mathcal{A})} \equiv 0 \text{ and } \Delta|_{\mathrm{sp}\{e_{i,j}\}_{i,j=1}^n} \equiv 0,$$
then it is identically equal to zero.

Below in the five Lemmata we shall consider 2-local derivations which satisfy the latter equalities.

We denote by e the unit of the algebra $\mathcal{A}$.

Lemma 2.9. *Let $x \in M_n(\mathcal{A})$. Then $\Delta(x)_{k,k} = 0$ for all $k \in \overline{1,n}$.*

Proof. Let $x \in M_n(\mathcal{A})$, and fix $k \in \overline{1,n}$. Since Δ is homogeneous, we can assume that $\|x_{k,k}\| < 1$, where $\|\cdot\|$ is the norm on $\mathcal{A}$. Take a diagonal element y in $M_n(\mathcal{A})$ with $y_{k,k} = e + x_{k,k}$ and $y_{i,i} = 0$ otherwise. Since $\|x_{k,k}\| < 1$, it follows that $e + x_{k,k}$ is invertible in $\mathcal{A}$. Take a derivation $D = \mathrm{ad}(a) + \overline{\delta}$ of the form (2.2) such that
$$\Delta(x) = [a, x] + \overline{\delta}(x),\ \Delta(y) = [a, y] + \overline{\delta}(y).$$
Since $y \in \mathcal{D}_n(\mathcal{A})$ we have that $0 = \Delta(y) = [a, y] + \overline{\delta}(y)$, and therefore
$$\begin{aligned} 0 &= \Delta(y)_{k,k} = a_{k,k}(e + x_{k,k}) - (e + x_{k,k})a_{k,k} + \delta(e + x_{k,k}) = 0, \\ 0 &= \Delta(y)_{i,k} = a_{i,k}(e + x_{k,k}) = 0, \\ 0 &= \Delta(y)_{k,i} = -(e + x_{k,k})a_{k,i} = 0 \end{aligned}$$
for all $i \neq k$. Thus
$$a_{k,k}x_{k,k} - x_{k,k}a_{k,k} + \delta(x_{k,k}) = 0$$
and
$$a_{i,k} = a_{k,i} = 0$$

for all $i \neq k$. The above equalities imply that

$$\Delta(x)_{k,k} = a_{k,k}x_{k,k} - x_{k,k}a_{k,k} + \delta(x_{k,k}) = \Delta(y)_{k,k} = 0.$$

The proof is complete. $\square$

Lemma 2.10. *Let x be a matrix with $x_{k,s} = e$. Then $\Delta(x)_{k,s} = 0$.*

Proof. We have

$$\begin{array}{rcl} e_{s,k}\Delta(x)e_{s,k} & = & e_{s,k}D_{e_{s,k},x}(x)e_{s,k} \\ & = & D_{e_{s,k},x}(e_{s,k}xe_{s,k}) - D_{e_{s,k},x}(e_{s,k})xe_{s,k} - e_{s,k}xD_{e_{s,k},x}(e_{s,k}) \\ & = & D_{e_{s,k},x}(e_{s,k}) - \Delta(e_{s,k})xe_{s,k} - e_{s,k}x\Delta(e_{s,k}) \\ & = & \Delta(e_{s,k}) - 0 - 0 = 0. \end{array}$$

Thus

$$e_{k,k}\Delta(x)e_{s,s} = e_{k,s}e_{s,k}\Delta(x)e_{s,k}e_{k,s} = 0.$$

This means that $\Delta(x)_{k,s} = 0$. The proof is complete. $\square$

Lemma 2.11. *Let k, s be numbers such that $k \neq s$ and let x be a matrix with $x_{k,s} = e$. Then $\Delta(x)_{s,k} = 0$.*

Proof. Take a diagonal element y such that $y_{k,k} = x_{s,k}$ and $y_{i,i} = \lambda_i e$ otherwise, where λ_i $(i \neq k)$ are distinct numbers with $|\lambda_i| > \|x_{s,k}\|$. Take a derivation $D = \mathrm{ad}(a) + \overline{\delta}$ such that

$$\Delta(x) = [a, x] + \overline{\delta}(x) \text{ and } \Delta(y) = [a, y] + \overline{\delta}(y).$$

Then

$$\begin{array}{rcl} 0 & = & \Delta(y)_{ij} = \lambda_j a_{i,j} - \lambda_i a_{i,j} = a_{i,j}(\lambda_j - \lambda_i),\ i \neq j,\ (i-k)(j-k) \neq 0, \\ 0 & = & \Delta(y)_{i,k} = a_{i,k}y_{k,k} - \lambda_i a_{i,k} = a_{i,k}(x_{s,k} - \lambda_i),\ i \neq k, \\ 0 & = & \Delta(y)_{k,j} = a_{k,j}\lambda_j - y_{kk}a_{kj} = (\lambda_j - x_{s,k})a_{k,j},\ j \neq k. \end{array}$$

Thus $a_{i,j} = 0$ for all $i \neq j$, i.e. a is a diagonal element. Since

$$0 = \Delta(x)_{ks} = a_{kk} - a_{ss},$$

it follows that $a_{k,k} = a_{s,s}$. Finally,

$$\begin{array}{rcl} \Delta(x)_{s,k} & = & a_{s,s}x_{s,k} - x_{s,k}a_{k,k} + \delta(x_{s,k}) \\ & = & a_{k,k}x_{s,k} - x_{s,k}a_{k,k} + \delta(y_{k,k}) = \Delta(y)_{k,k} = 0. \end{array}$$

The proof is complete. $\square$

Lemma 2.12. *Let $k \neq s$ and let x, y be matrices with $x_{i,j} = y_{i,j}$ for all $(i, j) \neq (s, k)$. Then $\Delta(x)_{k,s} = \Delta(y)_{k,s}$.*

Proof. Take a derivation $D = \mathrm{ad}(a) + \overline{\delta}$ such that

$$\Delta(x) = [a, x] + \overline{\delta}(x) \text{ and } \Delta(y) = [a, y] + \overline{\delta}(y).$$

Then

$$\begin{aligned}\Delta(x)_{k,s} &= \sum_{j=1}^{n}(a_{k,j}x_{j,s}-x_{k,j}a_{j,s})+\delta(x_{ks})\\ &= \sum_{j=1}^{n}(a_{k,j}y_{j,s}-y_{k,j}a_{j,s})+\delta(y_{ks})=\Delta(y)_{k,s}.\end{aligned}$$

The proof is complete. □

Lemma 2.13. *Let $k \neq s$. Then $\Delta(x)_{k,s}=0$.*

Proof. Take a matrix y with $y_{s,k}=e$ and $y_{i,j}=x_{i,j}$ otherwise. By Lemma 2.11 we have that $\Delta(y)_{k,s}=0$. Further Lemma 2.12 implies that

$$\Delta(x)_{k,s}=\Delta(y)_{k,s}=0.$$

The proof is complete. □

Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1. Let Δ be a 2-local derivation from $M_n(\mathcal{A})$ into $M_n(\mathcal{M})$, where $n\geq 3$. By Lemma 2.2 there exists a derivation D such that $\Delta|_{\mathrm{sp}\{e_{i,j}\}_{i,j=1}^n} = D|_{\mathrm{sp}\{e_{i,j}\}_{i,j=1}^n}$. Consider a 2-local derivation $\Theta=\Delta-D$. Since Θ is equal to zero on $\mathrm{sp}\{e_{i,j}\}_{i,j=1}^n$, by Lemma 2.8 we obtain that $\Theta|_{\mathcal{D}_n(\mathcal{A})}=\overline{\Theta_{11}}|_{\mathcal{D}_n(\mathcal{A})}$, where $\overline{\Theta_{11}}$ is the derivation defined by (2.1). As in Lemma 2.8 we have that

$$\left(\Theta-\overline{\Theta_{11}}\right)|_{\mathrm{sp}\{e_{i,j}\}_{i,j=1}^n}\equiv 0 \text{ and } \left(\Theta-\overline{\Theta_{11}}\right)|_{\mathcal{D}_n(\mathcal{A})}\equiv 0.$$

Now for an arbitrary element $x\in M_n(\mathcal{A})$, by Lemmata 2.9 and 2.13 we obtain that $\left(\Theta-\overline{\Theta_{11}}\right)(x)_{k,s}=0$ for all k,s. Thus $\left(\Theta-\overline{\Theta_{11}}\right)(x)=0$, i.e., $\Theta=\overline{\Theta_{11}}$. So, $\Delta=\overline{\Theta_{11}}+D$ is a derivation. The proof is complete. □

3. An application to 2-local derivations on algebras of locally measurable operators

In this section we apply Theorem 2.1 to the description of 2-local derivations on the algebra of locally measurable operators affiliated with a von Neumann algebra and on its subalgebras.

In [8, Corollary 3.11] it was proved that if an associative algebra (ring) $\mathcal{A}$ contains a noncommutative simple subalgebra (subring) $\mathcal{A}_0$ which contains the unit of $\mathcal{A}$, then every Jordan derivation from $\mathcal{A}$ into any $\mathcal{A}$-bimodule is a derivation, i.e. $\mathcal{A}$ satisfies the property **(J)**. In particular, if there exists a subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ which is isomorphic to $M_n(\mathbb{C})$ $(n\geq 2)$ and contains the unit of $\mathcal{A}$, then $\mathcal{A}$ has the property **(J)**.

Let M be a von Neumann algebra and denote by $S(M)$ the algebra of all measurable operators and by $LS(M)$ the algebra of all locally measurable operators affiliated with M (see for example [16, 18]).

Theorem 3.1. *Let M be an arbitrary von Neumann algebra without abelian direct summands and let $LS(M)$ be the algebra of all locally measurable operators affiliated with M. Then any 2-local derivation Δ from M into $LS(M)$ is a derivation.*

Proof. Let z be a central projection in M. Since $D(z) = 0$ for an arbitrary derivation D, it is clear that $\Delta(z) = 0$ for any 2-local derivation Δ from M into $LS(M)$. Take $x \in M$ and let D be a derivation from M into $LS(M)$ such that $\Delta(zx) = D(zx), \Delta(x) = D(x)$. Then we have $\Delta(zx) = D(zx) = D(z)x + zD(x) = z\Delta(x)$. This means that every 2-local derivation Δ maps zM into $zLS(M) \cong LS(zM)$ for each central projection $z \in M$. So, we may consider the restriction of Δ onto zM. Since an arbitrary von Neumann algebra without abelian direct summands can be decomposed along a central projection into the direct sum of von Neumann algebras of type $\mathrm{I}_n, n \geq 2$, type I_∞, type II and type III, we may consider these cases separately.

If M is a von Neumann algebra of type I_n, $n \geq 2$, [10, Corollary 3.12] implies that any 2-local derivation from M into $LS(M) \equiv S(M)$ is a derivation.

Let the von Neumann algebra M have one of the types I_∞, II or III. Then the halving Lemma [13, Lemma 6.3.3] for type I_∞-algebras and [13, Lemma 6.5.6] for type II or III algebras, imply that the unit of the algebra M can be represented as a sum of mutually equivalent orthogonal projections e_1, e_2, e_3 from M. Then the map $x \mapsto \sum\limits_{i,j=1}^{3} e_i x e_j$ defines an isomorphism between the algebra M and the matrix algebra $M_3(\mathcal{A})$, where $\mathcal{A} = e_{1,1} M e_{1,1}$. Further, the algebra $LS(M)$ is isomorphic to the algebra $M_3(LS(\mathcal{A}))$. Moreover, the algebra $\mathcal{A}$ has same type as the algebra M, and therefore contains a subalgebra isomorphic to $M_3(\mathbb{C})$. This means that the algebra $\mathcal{A}$ satisfies the property **(J)**. Therefore Theorem 2.1 implies that any 2-local derivation from M into $LS(M)$ is a derivation. The proof is complete. $\square$

Taking into account that any derivation on an abelian von Neumann algebra is trivial, Theorem 3.1 implies the following result (cf. [2, Theorem 2.1] and [3, Theorem 3.1]).

Corollary 3.2. *Let M be an arbitrary von Neumann algebra. Then any 2-local derivation Δ on M is a derivation.*

For each $x \in LS(M)$ set $s(x) = l(x) \vee r(x)$, where $l(x)$ is the left and $r(x)$ is the right support of x.

Lemma 3.3. *Let $\mathcal{B}$ be a subalgebra of $LS(M)$ such that $M \subseteq \mathcal{B}$ and let $\Delta : \mathcal{B} \to LS(M)$ be a 2-local derivation such that $\Delta|_M \equiv 0$. Then $\Delta \equiv 0$.*

Proof. Let us first take an arbitrary element $x \in \mathcal{B} \cap S(M)$. Let $|x| = \int\limits_0^\infty \lambda \, de_\lambda$ be the spectral resolution of $|x|$. Since $x \in S(M)$, it follows that $e_n^\perp$ is a finite projection for a sufficiently large n. Take a derivation D_{x,xe_n} such that $\Delta(x) = D_{x,xe_n}(x)$ and $\Delta(xe_n) = D_{x,xe_n}(xe_n)$, $n \in \mathbb{N}$. Since $xe_n \in M$, it follows that $\Delta(xe_n) = 0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} \Delta(x) &= \Delta(x) - \Delta(xe_n) = D_{x,xe_n}(x) - D_{x,xe_n}(xe_n) \\ &= D_{x,xe_n}(x - xe_n) = D_{x,xe_n}(xe_n^\perp). \end{aligned}$$

Let $\mathcal{D}$ be a dimension function on the lattice $P(M)$ of all projections from M (see [18]). Using [6, Lemma 4.3] we obtain that

$$\begin{aligned}\mathcal{D}(s(\Delta(x))) &= \mathcal{D}(s(D_{x,xe_n}(xe_n^\perp))) \leq 3\mathcal{D}(s(xe_n^\perp)) = 3\mathcal{D}(l(xe_n^\perp) \vee r(xe_n^\perp)) \\ &\leq 3\mathcal{D}\left(l(xe_n^\perp)\right) + 3\mathcal{D}(r(xe_n^\perp)) \leq 6\mathcal{D}(e_n^\perp) \downarrow 0,\end{aligned}$$

and therefore $\Delta(x) = 0$.

Now let take an element $x \in \mathcal{B}$. By the definition of locally measurable operator there exists a sequence $\{z_n\}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $xz_n \in S(M)$ for all $n \in \mathbb{N}$ (see [16]). Taking into account the previous case we obtain that

$$\begin{aligned}z_n\Delta(x) &= z_nD_{x,z_nx}(x) = D_{x,z_nx}(z_nx) - D_{x,z_nx}(z_n)x \\ &= D_{x,z_nx}(z_nx) = \Delta(z_nx) = 0,\end{aligned}$$

i.e., $z_n\Delta(x) = 0$ for all $n \in \mathbb{N}$. Hence $\Delta(x) = 0$. The proof is complete. □

Theorem 3.4. *(cf. [4, Theorem 5.5]). Let M be an arbitrary von Neumann algebra without abelian direct summands and let $\mathcal{B}$ be a subalgebra of $LS(M)$ such that $M \subseteq \mathcal{B}$. Then any 2-local derivation Δ on $\mathcal{B}$ is a derivation.*

Proof. By Theorem 3.1 the restriction $\Delta|_M$ of Δ, is a derivation from M into $LS(M)$. By [6, Theorem 4.8] the derivation $\Delta|_M$ can be extended to a derivation from $\mathcal{B}$ into $LS(M)$, which we denote by D. Since the 2-local derivation $\Delta - D$ is equal to zero on M, Lemma 3.3 implies that $\Delta \equiv D$. The proof is complete. □

Remark 3.5. As it was mentioned in the introduction, the paper [5] gives necessary and sufficient conditions on a commutative regular algebra to admit 2-local derivations which are not derivations. In particular, for an arbitrary abelian von Neumann algebra M with a non atomic lattice of projections $P(M)$ the algebras $S(M)$ and $LS(M)$ always admit a 2-local derivation which is not a derivation.

A complete description of derivations on the algebra $LS(M)$ for type I von Neumann algebras M is given in [4, Section 3]). Moreover, for general von Neumann algebras every derivation on the algebra $LS(M)$ is inner, provided that M is a properly infinite von Neumann algebra [4, 7]. But for type II_1 von Neumann algebra M description of structure of derivations on the algebra $S(M) \equiv LS(M)$ is still an open problem (see [4]). In this connection it should be noted that Theorem 3.4 is one of the first results on 2-local derivations without information on the general form of derivations on these algebras.

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