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(p, q)-TYPE BETA FUNCTIONS OF SECOND KIND

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ABSTRACT. In the present article, we propose the (p, q)-variant of beta function of second kind and establish a relation between the generalized beta and gamma functions using some identities of the post-quantum calculus. As an application, we also propose the (p, q)-Baskakov–Durrmeyer operators, estimate moments and establish some direct results.

1. INTRODUCTION

The quantum calculus (q-calculus) in the field of approximation theory was discussed widely in the last two decades. Several generalizations to the q variants were recently presented in the book [3]. Further there is possibility of extension of q-calculus to post-quantum calculus, namely the (p, q)-calculus. Actually such extension of quantum calculus can not be obtained directly by substitution of q by q/p in q-calculus. But there is a link between q-calculus and (p, q)-calculus. The q calculus may be obtained by substituting $p = 1$ in (p, q)-calculus. We mentioned some previous results in this direction. Recently, Gupta [8] introduced (p, q) genuine Bernstein–Durrmeyer operators and established some direct results. (p, q) generalization of Szász–Mirakyan operators was defined in [1]. Also authors investigated a Durrmeyer type modifications of the Bernstein operators in [9]. We can also mention other papers as Bernstein operators [10], Bernstein–Stancu operators [11]. Bleimann–Butzer–Hahn operators and Szász–Mirakyan–Kantorovich

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operators . Besides this, we also refer to some recent related work on this topic: e.g. [5], [12] and [13].

Some basic notations of (p,q)-calculus are mentioned below:

The (p,q)-numbers are defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Obviously, it may be seen that $[n]_{p,q} = p^{n-1}[n]_{q/p}$. In The (p,q)-factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, n \geq 1, [0]_{p,q}! = 1.$$

The (p,q)-binomial coefficient is given by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! \, [k]_{p,q}!}, 0 \leq k \leq n.$$

For details see [15] and [16].

Definition 1.1. The (p,q)-power basis is defined below and it also has a link with q-power basis as

$$(x \oplus a)_{p,q}^n = (x+a)(px+qa)(p^2x+q^2a)\cdots(p^{n-1}x+q^{n-1}a).$$

$$(x \ominus a)_{p,q}^n = (x-a)(px-qa)(p^2x-q^2a)\cdots(p^{n-1}x-q^{n-1}a).$$

Definition 1.2. The (p,q)-derivative of the function f is defined as

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0$$

and $D_{p,q} f(0) = f'(0)$, provided that f is differentiable at 0. Note also that for $p = 1$, the (p,q)-derivative reduces to the $q-$derivative. The (p,q)-derivative fulfils the following product rules

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)$$
$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

The following assertions hold true:

$$D_{p,q}(x \ominus a)_{p,q}^n = [n]_{p,q}(px \ominus a)_{p,q}^{n-1}, n \geq 1$$
$$D_{p,q}(a \ominus x)_{p,q}^n = -[n]_{p,q}(a \ominus qx)_{p,q}^{n-1}, n \geq 1,$$

and $D_{p,q}(x \ominus a)_{p,q}^0 = 0$.

Definition 1.3. ([14])Let n is a nonnegative integer, we define the (p,q)-gamma function as

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)_{p,q}^n}{(p-q)^n} = [n]_{p,q}!, \quad 0 < q < p.$$

Proposition 1.4. *The formula of (p,q)-integration by part is given by*

$$\int_a^b f(px)\, D_{p,q}g(x)\, d_{p,q}x = f(b)\,g(b) - f(a)\,q(a) - \int_a^b g(qx)\, D_{p,q}f(x)\, d_{p,q}x \quad (1.1)$$

In the present paper, we propose the (p,q)-Baskakov–Durrmeyer operators and estimate some approximation properties, which include asymptotic formula and convergence in terms of modulus of continuity.

2. (p,q)-beta Function of Second Kind

Let $m,n \in \mathbb{N}$, we define (p,q)-beta function of second kind as

$$B_{p,q}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1\oplus px)_{p,q}^{m+n}} d_{p,q}x$$

Theorem 2.1. *Let $m,n \in \mathbb{N}$. We have the following relation between (p,q)-beta and (p,q)-gamma function:*

$$B_{p,q}(m,n) = q^{[2-m(m-1)]/2} p^{-m(m+1)/2} \frac{\Gamma_{p,q}(m)\,\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

Proof. We know that

$$D_{p,q}\frac{1}{(1\oplus x)_{p,q}^{n}} = -\frac{p\,[n]_{p,q}}{(1\oplus px)_{p,q}^{n+1}}$$

If we choose $f(x) = x^m$ and $g(x) = -\frac{1}{p[m+n]_{p,q}(1\oplus x)_{p,q}^{m+n}}$ and use (1.1) we have

$$\begin{aligned}
B_{p,q}(m+1,n) &= \int_0^\infty \frac{x^m}{(1\oplus px)_{p,q}^{m+n+1}} d_{p,q}x \\
&= -\frac{p^{-m}}{p\,[m+n]_{p,q}} \int_0^\infty (px)^m\, D_{p,q}\frac{1}{(1\oplus x)_{p,q}^{m+n}} d_{p,q}x \\
&= \frac{p^{-m}}{p\,[m+n]_{p,q}} \int_0^\infty D_{p,q}x^m \frac{1}{(1\oplus qx)_{p,q}^{m+n}} d_{p,q}x \\
&= \frac{p^{-m}\,[m]_{p,q}}{p\,[m+n]_{p,q}} \int_0^\infty x^{m-1} \frac{1}{(1\oplus qx)_{p,q}^{m+n}} d_{p,q}x \\
&= \frac{p^{-m-1}\,[m]_{p,q}}{q^{m-1}\,[m+n]_{p,q}} \int_0^\infty (qx)^{m-1} \frac{1}{(1\oplus qx)_{p,q}^{m+n}} d_{p,q}x \\
&= \frac{p^{-1}\,[m]_{p,q}}{(pq)^m\,[m+n]_{p,q}} \int_0^\infty (x)^{m-1} \frac{1}{(1\oplus x)_{p,q}^{m+n}} d_{p,q}x \\
&= \frac{p^{-1}\,[m]_{p,q}}{(pq)^m\,[m+n]_{p,q}} B_{p,q}(m,n),
\end{aligned}$$

$$B_{p,q}(1,n) = \int_0^\infty \frac{1}{(1\oplus px)_{p,q}^{n+1}} d_{p,q}x = -\frac{1}{p\,[n]_{p,q}} \int_0^\infty D_{p,q}\frac{1}{(1\oplus x)_{p,q}^{n}} d_{p,q}x = \frac{1}{p\,[n]_{p,q}}$$

and

$$\begin{aligned}
&B_{p,q}(m,n)\\
&=\frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}}B_{p,q}(m-1,n)\\
&=\frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}}\frac{p^{-1}[m-2]_{p,q}}{(pq)^{m-2}[m+n-2]_{p,q}}B_{p,q}(m-2,n)\\
&=\frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}}\frac{p^{-1}[m-2]_{p,q}}{(pq)^{m-2}[m+n-2]_{p,q}}\cdots\frac{p^{-1}}{pq[n+1]_{p,q}}B_{p,q}(1,n)\\
&=\frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}}\frac{p^{-1}[m-2]_{p,q}}{(pq)^{m-2}[m+n-2]_{p,q}}\cdots\frac{p^{-1}}{pq[n+1]_{p,q}}\frac{q}{pq[n]_{p,q}}\\
&=\frac{qp^{-m}}{(pq)^{(m-1)m/2}}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}
\end{aligned}$$

□

3. (p,q)-Baskakov–Durrmeyer Operators and Moments

The (p,q)-analogue of Baskakov operators for $x\in[0,\infty)$ and $0<q<p\leq 1$ is defined as

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n}b_{n,k}^{p,q}(x)f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \tag{3.1}$$

where

$$b_{n,k}^{p,q}(x)=\left[\begin{array}{c} n+k-1\\ k\end{array}\right]_{p,q}p^{k+n(n-1)/2}q^{k(k-1)/2}\frac{x^k}{(1\oplus x)_{p,q}^{n+k}}.$$

In case $p=1$, we get the q-Baskakov operators [2]. If $p=q=1$, we get at once the well known Baskakov operators.

Remark 3.1. Starting with the following relations between (p,q)-calculus and q-calculus:

$$\left[\begin{array}{c} n+k-1\\ k\end{array}\right]_{p,q}=p^{k(n-1)}\left[\begin{array}{c} n+k-1\\ k\end{array}\right]_{q/p}$$

and

$$(x\oplus a)_{p,q}^n=p^{n(n-1)/2}(x+a)_{q/p}^n$$

and using moments of q-Baskakov operators (see [2], [3]), it can easily be verified by simple computation that

$$B_{n,p,q}(1;x)=1, B_{n,p,q}(t;x)=x, B_{n,p,q}\left(t^2;x\right)=x^2+\frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right).$$

Definition 3.2. Using (p,q)-beta function of second kind, we propose below for $x \in [0,\infty), 0 < q < p \leq 1$ the (p,q) analogue of Baskakov–Durrmeyer operators

$$D_n^{p,q}(f;x) = [n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{t^k}{(1\oplus pt)_{p,q}^{k+n}} f(p^k t) d_{p,q}t \tag{3.2}$$

where $b_{n,k}^{p,q}(x)$ is as defined in (3.1).

Lemma 3.3. *For $x \in [0,\infty]$, $0 < q < p \leq 1$, we have*

(1) $D_n^{p,q}(1;x) = 1$

(2) $D_n^{p,q}(t;x) = \frac{1}{qp^2[n-2]_{p,q}} + \frac{[2]_{p,q}}{p^2q^2[n-2]_{p,q}}x + \frac{1}{p^n}x$

(3) $D_n^{p,q}(t^2;x) = \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \left(\frac{\left(p^5q(q+2p)+1\right)[3]_{p,q}}{p^6q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{p^5q(q+2p)+1}{p^{3+n}q[n-2]_{p,q}}\right)x + \frac{q^2+pq+p^2}{p^{9+n}q^2[n-2]_{p,q}}x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}}x^2 + \frac{\left(p^{n+2}[3]_{p,q}+q[2]_{p,q}[3]_{p,q}\right)}{p^{12+n}q^6[n-2]_{p,q}[n-3]_{p,q}}x^2 + \frac{1}{p^{7+2n}}x^2.$

Proof. Using (3.2) and Remark 3.1, we have

$$\begin{aligned} D_n^{p,q}(1;x) &= [n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\ &\quad \times \int_0^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{t^k}{(1\oplus pt)_{p,q}^{k+n}} d_{p,q}t \\ &= [n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\ &\quad \times \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} B_{p,q}(k+1,n-1) \\ &= B_{n,p,q}(1;x) = 1. \end{aligned}$$

Next using the identity $[k+1]_{p,q} = q^k + p[k]_{p,q}$ and applying Remark 3.1, we have

$$\begin{aligned}
&D_n^{p,q}(t;x)\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2}\int_0^\infty \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{t^{k+1}p^k}{(1\oplus pt)_{p,q}^{k+n}} d_{p,q}t\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty} q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} b_{n,k}^{p,q}(x) \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} p^k B_{p,q}(k+2, n-2)\\
&= \sum_{k=0}^{\infty} p^{-2} q^{-k-1} b_{n,k}^{p,q}(x) . \frac{[k+1]_{p,q}}{[n-2]_{p,q}}\\
&= \frac{1}{[n-2]_{p,q}p^2}\sum_{k=0}^{\infty} q^{-k-1} b_{n,k}^{p,q}(x)(q^k + p[k]_{p,q})\\
&= \frac{1}{[n-2]_{p,q}qp^2} B_{n,p,q}(1;x) + \frac{[n]_{p,q}}{p^n q^2 [n-2]_{p,q}} B_{n,p,q}(t;x)\\
&= \frac{1}{qp^2[n-2]_{p,q}} + \frac{[n]_{p,q}x}{p^n q^2 [n-2]_{p,q}}.
\end{aligned}$$

Further using the identity $[k+2]_{p,q} = q^{k+1} + pq^k + p^2[k]_{p,q}$ and by Remark 3.1, we get

$$\begin{aligned}
D_n^{p,q}(t^2;x) &= [n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2}\\
&\quad\times \int_0^\infty \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} \frac{t^{k+2}p^{2k}}{(1\oplus pt)_{p,q}^{k+n}} d_{p,q}t\\
&= [n-1]_{p,q}\sum_{k=0}^{\infty} q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} b_{n,k}^{p,q}(x)\\
&\quad\times \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} p^{2k} . B_{p,q}(k+3, n-3)\\
&= \sum_{k=0}^{\infty} q^{-(2k+3)} p^{-5} b_{n,k}^{p,q}(x) . \frac{[k+2]_{p,q}[k+1]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}}\\
&= \sum_{k=0}^{\infty} \frac{b_{n,k}^{p,q}(x) p^{-5} . q^{-(2k+3)} \left(p^3[k]_{p,q}^2 + q^k\left(p[2]_{p,q} + p^2\right)[k]_{p,q} + q^{2k}[2]_{p,q}\right)}{[n-2]_{p,q}[n-3]_{p,q}}\\
&= \frac{1}{[n-2]_{p,q}[n-3]_{p,q}}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)\Bigg[\left(\frac{p^{n-1}}{q^{k-1}}[k]_{p,q}\right)^2 \frac{p^{-7-2n}}{q^5}\\
&\quad + ([2]_{p,q} + p)\left(\frac{p^{n-1}}{q^{k-1}}[k]_{p,q}\right)\frac{p^{-3-n}}{q^4} + q^{-3}[2]_{p,q}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^{-3}[2]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}(1;x) + \frac{p^{-3-n}}{q^4}\frac{([2]_{p,q}+p)\,[n]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}(t;x) \\
&\quad + \frac{p^{-7-2n}}{q^5}\frac{[n]_{p,q}^2}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}\left(t^2;x\right) \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q}+p)\,[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{[n]_{p,q}^2}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} \left(x^2 + \frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right) \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q}+p)\,[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{[n]_{p,q}^2}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} \left(x^2 + \frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q}+p)\,[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{[n]_{p,q}^2}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \\
&\quad \times \frac{[n]_{p,q}}{p^{7+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2. \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q}+p)\,[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{1}{p^{7+2n}} x^2 + \frac{[2]_{p,q}}{p^{9+n}q^2[n-2]_{p,q}} x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}} x^2 \\
&\quad + \frac{[2]_{p,q}[3]_{p,q}}{p^{12+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 \\
&\quad + \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \frac{[n]_{p,q}}{p^{7+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \left(\frac{(p^5q\,(q+2p)+1)\,[3]_{p,q}}{p^6q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{p^5q\,(q+2p)+1}{p^{3+n}q[n-2]_{p,q}}\right) x \\
&\quad + \frac{q^2+pq+p^2}{p^{9+n}q^2[n-2]_{p,q}} x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}} x^2 \\
&\quad + \frac{(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q})}{p^{12+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{1}{p^{7+2n}} x^2.
\end{aligned}
$$

□

4. Weighted approximation

We consider the following class of functions:

Let $H_{x^2}[0,\ \infty)$ be the set of all functions f defined on $[0,\ \infty)$ satisfying the condition $|f(x)| \leq M_f(1+x^2)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\ \infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\ \infty)$. Also, let $C^*_{x^2}[0,\ \infty)$ be the subspace of all functions $f \in C_{x^2}[0,\ \infty)$, for which $\lim\limits_{|x|\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\ \infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\ \infty)} \frac{|f(x)|}{1+x^2}$.

Now we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval $[0,\ \infty)$.

Theorem 4.1. *Let $p = p_n$ and $q = q_n$ satisfies $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \to 1$, $q_n \to 1$ and $q_n^n \to 1$ and $p_n^n \to 1$. For each $f \in C^*_{x^2}[0,\ \infty)$, we have*

$$\lim_{n\to\infty} \|D_n^{p_n,q_n}(f) - f\|_{x^2} = 0.$$

Proof. Using the Theorem in [7] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|D_n^{p_n,q_n}(t^\nu, x) - x^\nu\|_{x^2} = 0, \quad \nu = 0,1,2. \tag{4.1}$$

Since $D_n^{p_n,q_n}(1,x) = 1$ the first condition of (4.1) is fulfilled for $\nu = 0$.

We can write for $n > 3$

$$\begin{aligned}\|D_n^{p_n,q_n}(t,x) - x\|_{x^2} &\leq \frac{1}{q_n p_n^2 [n-2]_{p_n,q_n}} \\ &\quad + \left(\frac{[2]_{p_n,q}}{p_n^n q_n^2 [n-2]_{p_n,q_n}} + \frac{1}{p_n^n} - 1\right) \sup_{x\in[0,\ \infty)} \frac{x}{1+x^2}\end{aligned}$$

and

$$\begin{aligned}&\left\|D_n^{p_n,q_n}(t^2,x) - x^2\right\|_{x^2} \\ \leq\ &\left(\frac{q_n^2 + p_n q_n + p_n^2}{p_n^{9+n} q_n^2 [n-2]_{p_n,q_n}} + \frac{[3]_{p_n,q_n}}{p_n^{10+n} q_n^3 [n-3]_{p,q}} + \frac{(p_n^{n+2}[3]_{p_n,q_n} + q_n[2]_{p_n,q_n}[3]_{p_n,q_n})}{p_n^{12+n} q_n^6 [n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\right) \\ &\times \sup_{x\in[0,\ \infty)} \frac{x^2}{1+x^2} \\ &+ \left(\frac{1}{p_n^{7+2n}} - 1\right) \sup_{x\in[0,\ \infty)} \frac{x^2}{1+x^2} \\ &+ \left(\frac{(p_n^5 q_n (q_n + 2p_n) + 1)[3]_{p_n,q_n}}{p_n^6 q_n^4 [n-2]_{p_n,q_n}[n-3]_{p_n,q_n}} + \frac{p_n^5 q_n (q_n + 2p_n) + 1}{p_n^{3+n} q_n [n-2]_{p_n,q_n}}\right) \sup_{x\in[0,\ \infty)} \frac{x}{1+x^2} \\ &+ \frac{[2]_{p_n,q_n}}{q_n^3 [n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|D_n^{p_n,q_n}(t,x) - x\|_{x^2} = 0$$

and

$$\lim_{n\to\infty}\left\|D_n^{p_n,q_n}\left(t^2,x\right)-x^2\right\|_{x^2}=0.$$

Thus the proof is completed. □

We give the following theorem to approximate all functions in $C_{x^2}[0,\ \infty)$.

Theorem 4.2. *Let $p=p_n$ and $q=q_n$ satisfies $0<q_n<p_n\leq 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $q_n^n\to 1$ and $p_n^n\to 1$. For each $f\in C_{x^2}^*[0,\ \infty)$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\ \infty)}\frac{\left|D_n^{p_n,q_n}(f,x)-f(x)\right|}{\left(1+x^2\right)^{1+\alpha}}=0.$$

Proof. For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\ \infty)}&\frac{\left|D_n^{p_n,q_n}(f,x)-f(x)\right|}{\left(1+x^2\right)^{1+\alpha}}\\&\leq\sup_{x\leq x_0}\frac{\left|D_n^{p_n,q_n}(f,x)-f(x)\right|}{\left(1+x^2\right)^{1+\alpha}}+\sup_{x\geq x_0}\frac{\left|D_n^{p_n,q_n}(f,x)-f(x)\right|}{\left(1+x^2\right)^{1+\alpha}}\\&\leq\left\|D_n^{p_n,q_n}(f)-f\right\|_{C[0,\ x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{\left|D_n^{p_n,q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}}\\&\quad+\sup_{x\geq x_0}\frac{|f(x)|}{\left(1+x^2\right)^{1+\alpha}}.\end{aligned}$$

The first term of the above inequality tends to zero from well known Korovkin's theorem. By Lemma 3.3 for any fixed $x_0>0$ it is easily seen that $\sup_{x\geq x_0}\frac{\left|D_n^{p_n,q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}}$ tends to zero as $n\to\infty$. We can choose $x_0>0$ so large that the last part of above inequality can be made small enough. □

Remark 4.3. For $q\in(0,1)$ and $p\in(q,1]$ it is seen that $\lim_{n\to\infty}[n]_{p,q}=1/(q-p)$. In order to consider convergence of (p,q) Baskakov operators we assume $p=(p_n)$ and $q=(q_n)$ such that $0<q_n<p_n\leq 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $p_n^n\to 1$ and $q_n^n\to 1$.

5. Quantitative approximation

Let $C_B[0,\infty)$ denote the space of all real valued continuous and bounded functions on $[0,\infty)$. In this space we consider the norm

$$\|f\|_{C_B}=\sup_{x\in[0,\infty)}|f(x)|.$$

Now we give the first and second order modulus of continuity of function $f\in C_B$ (see [4], [6]) The first modulus of continuity is defined as

$$\omega_1(f;\delta)=\sup_{\substack{x,u,v\geq 0\\|u-v|\leq\delta}}|f(x+u)-f(x+v)|$$

and the second order modulus of continuity is defined

$$\omega_2(f;\delta) = \sup_{\substack{x,u,v\geq 0\\ |u-v|\leq\delta}} |f(x+2u) - 2f(x+u+v+) f(x+2v)|, \quad \delta \geq 0.$$

We will use the Steklov mean function for $f \in C_B$

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v))]\, dudv. \tag{5.1}$$

Since $f_h \in C_B$ we can write

$$f_h(x) - f(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v)) - f(x)]\, dudv.$$

It is obvious that

$$|f_h(x) - f(x)| \leq \omega_2(f;h)$$

and

$$\|f_h - f\|_{C_B} \leq \omega_2(f;h). \tag{5.2}$$

If f is continuous, then $f_h' \in C_B$ and

$$\begin{aligned} f_h'(x) &= \frac{4}{h^2}\left[2\int_0^{\frac{h}{2}}\left(f\left(x+v+\frac{h}{2}\right) - f(x+v)\right) dv \right. \\ &\quad \left. -\frac{1}{2}\int_0^{\frac{h}{2}} (f(x+h+2v) - f(x+v))\, dv\right]. \end{aligned}$$

Thus we have

$$\|f_h'\|_{C_B} \leq \frac{5}{h}\omega_1(f;h). \tag{5.3}$$

Similarly $f_h'' \in C_B$ and

$$\|f_h''\|_{C_B} \leq \frac{9}{h^2}\omega_2(f;h). \tag{5.4}$$

Theorem 5.1. *Let $q \in (0,1)$ and $p \in (q,1]$. The operator $D_n^{p,q}$ maps space C_B into C_B and*

$$\|D_n^{p,q}(f)\|_{C_B} \leq \|f\|_{C_B}.$$

Proof. Let $q\in(0,1)$ and $p\in(q,1]$. From Lemma 3.3. we have

$$\begin{aligned}
\left|D_n^{p,q}(f;x)\right| &\leq [n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{[k(k+1)-2]/2}p^{(k+1)(k+2)/2} \\
&\quad \int_0^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}\frac{t^k}{(1\oplus pt)_{p,q}^{k+n}}\left|f(p^k t)\right| d_{p,q}t \\
&\leq \sup_{x\in[0,\infty)}|f(x)|\,[n-1]_{p,q}\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{[k(k+1)-2]/2}p^{(k+1)(k+2)/2} \\
&\quad \int_0^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q}\frac{t^k}{(1\oplus pt)_{p,q}^{k+n}} d_{p,q}t \\
&= \sup_{x\in[0,\infty)}|f(x)|\, D_n^{p,q}(1;x)=\|f\|_{C_B}.
\end{aligned}$$

□

We are going to study the degree of approximation in terms of $\omega_1(f;\delta)$ and $\omega_2(f;\delta)$, first and second order modulus of continuity.

Theorem 5.2. *Let $q\in(0,1)$ and $p\in(q,1]$. If $f\in C_B$, then*

$$\begin{aligned}
& \left|D_n^{p,q}(f;x)-f(x)\right| \\
\leq\; & 5\omega_1\left(f;\frac{1}{\sqrt{[n-2]_{p,q}}}\right) \\
& \times\left(\frac{1}{qp^2\sqrt{[n-2]_{p,q}}}+\frac{[2]_{p,q}}{p^2q^2\sqrt{[n-2]_{p,q}}}x+\left(\frac{1}{p^n}-1\right)\sqrt{[n-2]_{p,q}x}\right) \\
& +\frac{9}{2}\omega_2\left(f;\frac{1}{\sqrt{[n-2]_{p,q}}}\right)\left(\frac{p^{7+2n}-2p^{7+n}-1}{p^{7+2n}}\right)[n-2]_{p,q}x^2 \\
& +\frac{q^2+pq+p^2-2p^{8+n}-2qp^{7+n}}{p^{9+n}q^2}x^2 \\
& ++\frac{[3]_{p,q}[n-2]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}}x^2+\frac{(p^{n+2}[3]_{p,q}+q[2]_{p,q}[3]_{p,q})}{p^{12+n}q^6[n-3]_{p,q}}x^2 \\
& +\left(\frac{(p^5q(q+2p)+1)[3]_{p,q}}{p^6q^4[n-3]_{p,q}}+\frac{p^5q(q+2p)+1}{p^{3+n}q}-\frac{2}{qp^2}\right)x+\left.\frac{[2]_{p,q}}{q^3[n-3]_{p,q}}\right]
\end{aligned}$$

Proof. We use the Stieklov function f_h defined by (5.1). For $x\geq 0$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}
\left|D_n^{p,q}(f;x)-f(x)\right| &\leq D_n^{p,q}(|f-f_h|;x)+\left|D_n^{p,q}(f_h-f_h(x);x)\right| \\
&\quad +\left|f_h(x)-f(x)\right|.
\end{aligned}$$

By (5.2) we can write

$$D_n^{p,q}(|f-f_h|;x)\leq\left\|D_n^{p,q}(f-f_h)\right\|_{C_B}\leq\|f-f_h\|_{C_B}\leq\omega_2(f;h).$$

Since $D_n^{p,q}$ is a linear positive operator we get

$$\left|D_n^{p,q}\left(f_h - f_h\left(x\right); x\right)\right| \leq \left|f_h^{'}\left(x\right)\right| D_n^{p,q}\left(t-x; x\right) + \frac{1}{2}\left\|f''\right\|_{C_B} D_n^{p,q}\left(\left(t-x\right)^2; x\right).$$

By Lemma 3.3, (5.3) and (5.4) we have

$$\begin{aligned}
&\left|D_n^{p,q}\left(f_h - f_h\left(x\right); x\right)\right| \\
&\quad \leq \frac{5}{h}\omega_1\left(f;h\right)\left(\frac{1}{qp^2[n-2]_{p,q}} + \frac{[2]_{p,q}}{p^2q^2[n-2]_{p,q}}x + \left(\frac{1}{p^n}-1\right)x\right) \\
&\quad\quad + \frac{9}{2h^2}\omega_2\left(f;h\right) D_n^{p,q}\left(\left(t-x\right)^2; x\right),
\end{aligned}$$

where

$$\begin{aligned}
&D_n^{p,q}\left(\left(t-x\right)^2; x\right) \\
&= \left(\frac{p^{7+2n} - 2p^{7+n} - 1}{p^{7+2n}}\right)x^2 + \frac{q^2 + pq + p^2 - 2p^{8+n} - 2qp^{7+n}}{p^{9+n}q^2[n-2]_{p,q}}x^2 \\
&\quad + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}}x^2 + \frac{\left(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q}\right)}{p^{12+n}q^6[n-2]_{p,q}[n-3]_{p,q}}x^2 \\
&\quad + \left(\frac{\left(p^5q\left(q+2p\right)+1\right)[3]_{p,q}}{p^6q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{p^5q\left(q+2p\right)+1}{p^{3+n}q[n-2]_{p,q}} - \frac{2}{qp^2[n-2]_{p,q}}\right)x \\
&\quad + \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}}
\end{aligned}$$

for $x \geq 0$, $h > 0$. Setting $h = \sqrt{\frac{1}{[n-2]_{p,q}}}$, we have desired result. □

Remark 5.3. From Theorem 5.2 we can say that that the order of approximation of $D_n^{p,q}\left(f;x\right)$ to $f\left(x\right)$ is at least as good as the order of approximation to $f(x)$ by classical Baskakov–Durrmeyer operators for any $x \in [0,\infty)$ as a depending on selection of q_n and p_n. If we choose p and q as in Remark 4.3, we have an approximation process with the aid of operator (3.2).

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